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ON SUMS INVOLVING RECIPROCALS OF CERTAIN LARGE ADDITIVE FUNCTIONS (II)

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Abstract. Let $\beta(n)=\sum_{p|n}p$ and $B(n)=\sum_{p^\alpha\|n}\alpha p$. Let $p(n)$ denote the largest prime factor of an integer $n\geq 2$. In the present paper we sharpen the asymptotic formula for the sum $\sum\limits_{2\leq n\leq x}B(n)/\beta(n)$ and we derive an asymptotic formula for the sum $\sum\limits_{2\leq n\leq x}(B(n)-\beta(n))/p(n)$.

1. Introduction and statement of results

Let $\beta(n)=\sum_{p|n}p$ and $B(n)=\sum_{p^\alpha\|n}\alpha p$.

In [**2**] it was proved that

$$\sum_{2\leq n\leq x}B(n)/\beta(n)=x+O\big(x\exp(-c_1(\log x\log_2 x)^{1/2})\big),\quad c_1>0,\tag{1.1}$$

and

$$\sum_{2\leq n\leq x}\beta(n)/B(n)=x+O\big(x\exp(-c_2(\log x\log_2 x)^{1/2})\big),\quad c_2>0.\tag{1.2}$$

The above results were slightly sharpened in [**6**]. Let us define $p(n)$ as the largest prime factor of $n\geq 2$, and $p(1)=1$. In [**3**] it was proved that

$$\sum_{n\leq x}1/p(n)=x\delta(x)\Big(1+O((\log_2 x/\log x)^{1/2})\Big),$$

where

$$\delta(x)=\int_2^\infty\rho\left(\frac{\log x}{\log t}\right)t^{-2}\,dt.$$

Here $\rho(u)$ is the so-called "Dickman function", which is the solution of the differential-difference equation $u\rho'(u)+\rho(u-1)=0$, $(u>1)$, with the initial

AMS Subject Classification (1980): Primary 10 H 15, (10 H 25) condition $\rho(u) = 1$, $(0 \le u \le 1)$, $\rho(u)$ continuous at $u = 1$. An approximation to $\rho(u)$ in terms of elementary functions is

$$\rho(u) = \exp\{-u(\log u + \log_2 u - 1 + \log_2 u/\log u + O(1/\log u))\}, \tag{1.3}$$

where $\log_2 u = \log\log u$. The asymptotic formula (1.3) was established by Hua [**5**] and de Bruijn [**1**], independently.

In [**8**], we proved that

$$\sum_{2\le n\le x} 1/\beta(n) = \big(D + O(\log_3^2 x/\log_2 x)\big)\sum_{n\le x} 1/p(n), \tag{1.4}$$

where $1/2 < D < 1$ is an absolute constant.

One of the aimes of the present paper is to provide sharpenings of (1.1) and (1.2). The results are contained in the following theorem.

Theorem 1.

$$\sum_{2\le n\le x}\frac{B(n)}{\beta(n)} = x + \frac{1}{2}D\log x\left(1 + O\left(\frac{\log_3^2 x}{\log_2 x}\right)\right)\sum_{n\le x}\frac{1}{p(n)}, \tag{1.5}$$

and

$$\sum_{2\le n\le x}\frac{\beta(n)}{B(n)} = x - \frac{1}{2}D\log x\left(1 + O\left(\frac{\log_3^2 x}{\log_2 x}\right)\right)\sum_{n\le x}\frac{1}{p(n)}, \tag{1.6}$$

where D is the same as in (1.4).

Moreover, in [**9**] we proved that

$$\sum_{2\le n\le x}\frac{1}{p^r(n)} = x\exp\left\{-(2r\log x\log_2 x)^{1/2}\left(1+\frac{\sqrt{r}}{2\sqrt{2}}\frac{\log_3 x}{\log_2 x}+O\left(\frac{1}{\log_2 x}\right)\right)\right\}, \tag{1.7}$$

and

$$\sum_{2\le n\le x}\frac{B(n)-\beta(n)}{p(n)} = x\exp\left\{-(2r\log x\log_2 x)^{1/2}\left(1+\frac{\sqrt{r}}{2\sqrt{2}}\frac{\log_3 x}{\log_2 x}+O\left(\frac{1}{\log_2 x}\right)\right)\right\}, \tag{1.8}$$

where $r > 0$ is arbitrary but fixed.

Another aim of the present paper is to provide sharpenings of (1.8). The result is:

Theorem 2.

$$\sum_{2\le n\le x}\frac{B(n)-\beta(n)}{p(n)} = \frac{1}{2}\log x\left(1 + O\left(\frac{\log_3 x}{\log_2 x}\right)\right)\sum_{n\le x}\frac{1}{p(n)}.$$

By Theorem 1 and 2, we have

$$\sum_{2\le n\le x} \frac{B(n)-\beta(n)}{\beta(n)} \sim D \sum_{n\le x} \frac{B(n)-\beta(n)}{p(n)}.$$

It seems interesting to compare the sums involving reciprocals of $\beta(n)$ with the sums involving reciprocals of $p(n)$ [**8**]:

$$\sum_{2\le n\le x} \frac{1}{\beta(n)} \sim D \sum_{n\le x} \frac{1}{p(n)}, \qquad \sum_{2\le n\le x} \frac{\omega(n)}{\beta(n)} \sim D \sum_{n\le x} \frac{\omega(n)}{p(n)},$$

and

$$\sum_{2\le n\le x} \frac{\Omega(n)-\omega(n)}{\beta(n)} \sim D \sum_{n\le x} \frac{\Omega(n)-\omega(n)}{p(n)},$$

where $\Omega(n)$ and $\omega(n)$ denote respectively the number of prime factors of n counted with and without multiplicities.

2. The necessary lemmas

Lemma 1 [**7**]. *Let*

$$L_1 = \exp\left[\left(\frac{1}{2}\log x \log_2 x\right)^{1/2}\left(1 - 2\frac{\log_3 x}{\log_2 x}\right)\right], \qquad \textit{and}$$
$$L_2 = \exp\left[\left(\frac{1}{2}\log x \log_2 x\right)^{1/2}\left(1 + 2\frac{\log_3 x}{\log_2 x}\right)\right].$$

Then we have

$$\sum_{n\le x} \frac{1}{p(n)} = \sum_{L_1<p\le L_2} \frac{1}{p}\Psi\left(\frac{x}{p}, p\right)\left(1 + O(\log^{-A} x)\right),$$

where $A > 0$ is arbitrary but fixed, and $\Psi(x,y)$ denotes the number of positive integers not exceeding x, all of whose prime factors do not exceed y.

Lemma 2 [**4**]. *For any fixed $\varepsilon > 0$ and $x \ge 3$, $\exp\{(\log_2 x)^{5/3+\varepsilon}\} \le y \le x$, we have uniformly*

$$\Psi(x,y) = x\rho(u)\left(1 + O\left(\frac{\log(u+1)}{\log y}\right)\right), \qquad u = \frac{\log x}{\log y}.$$

Lemma 3 [**8**]. *For any fixed $\varepsilon > 0$ and $1 \le d \le y$, $\exp\{(\log_2 x)^{5/3+\varepsilon}\} \le y \le x^{1/2}$ we have uniformly*

$$\Psi(x/d,y) = \Psi(x,y)d^{-\beta}\left(1 + O(1/u) + \left(\frac{\log(u+1)}{\log y}\right)\right), \tag{2.1}$$

where

$$\beta = \beta(x,y) = 1 - \frac{\xi(\log x/\log y)}{\log y};$$

here $\xi(u)$ *denotes the positive solution of the equation*

$$e^{\xi} = u\xi + 1, \qquad (u > 1), \tag{2.2}$$

and satisfies

$$\xi(u) = \log u + O(\log_2(u+2)), \quad u \to \infty. \tag{2.3}$$

Lemma 4 [**8**]. *For any fixed* $\varepsilon > 0$ *and*

$$1 \le d \le y, \quad \exp\{(\log_2 x)^{5/3+\varepsilon}\} \le y \le x,$$

we have uniformly

$$\Psi(x/d, y) \ll \Psi(x,y)d^{-\beta}.$$

3. Proofs of the Theorems

We shall only give a detailed proof of Theorem 1, since Theorem 2 may be obtained in a similar and simpler way.

By the definition of $B(n)$ and $\beta(n)$ we have (p, q denote primes):

$$\begin{aligned} W(x) := \sum_{2\le n\le x} \frac{B(n)-\beta(n)}{\beta(n)} &= \sum_{q^\alpha \le x} (\alpha-1)q \sum_{2\le n\le x,\, q^\alpha \| n} \frac{1}{\beta(n)} \\ &= \sum_{q^\alpha\le x} (\alpha-1)q \sum_{q<p_1\le x/q^\alpha} \; \sum_{m_1\le x/q^\alpha p_1,\, p(m_1)\le p_1,\, (q,m_1)=1} \frac{1}{q+\beta(m_1p_1)} \\ &\quad + O\bigg(\sum_{q^\alpha\le x}(\alpha-1)\Psi(xq^{-\alpha},q)\bigg) = W_1 + O(W_2), \quad \text{say}. \end{aligned} \tag{3.1}$$

It is evident that

$$W_1 = \sum_{p_1\le x} \sum_{\substack{q^\alpha \le x/p_1,\\ q<p_1}} (\alpha-1)q \sum_{\substack{m_1\le x/q^\alpha p_1\\ p(m_1)\le p_1,\,(q,m_1)=1}} \frac{1}{q+\beta(m_1p_1)}.$$

We may write

$$W_1 = \sum_{p_1\le z_1} + \sum_{z_1<p_1\le L_1} + \sum_{L_1<p_1\le L_2} + \sum_{L_2<p_1\le x} = W_3+W_4+W_5+W_6, \tag{3.2}$$

where $z_1 = \exp\{(1/10)(\log x\log_2 x)^{1/2}\}$, and L_1 and L_2 are defined in Lemma 1. Let $R = (\log x \log_3 x/\log_2 x)\sum_{n\le x} 1/p(n)$; we have

$$W_3 \le \sum_{p_1\le z_1} \frac{1}{p_1} \sum_{q^\alpha\le x/p_1,\, q<p_1} (\alpha-1)q\Psi(xq^{-\alpha}p_1^{-1}, p_1)$$

$$\ll \log^2 x \left(\sum_{p_1 \le z_1} \frac{1}{p_1} \right) \sum_{q \le z_1} q \Psi(xq^{-2}, z_1) \tag{3.3}$$
$$\ll x \exp\{-4(\log x \log_2 x)^{1/2}\} \ll R,$$

since by Lemma 2 and (1.3) we have $\Psi(xq^{-2}, z_1) \ll xq^{-2} \exp\{-4.5(\log x \log_2 x)^{1/2}\}$. Using Lemma 4 we have

$$W_4 \le \sum_{z_1 < p_1 \le L_1} \frac{1}{p_1} \sum_{q^\alpha \le x/p_1,\, q<p_1} (\alpha - 1) q \Psi(xq^{-\alpha} p_1^{-1}, p_1)$$
$$\ll \sum_{z_1 < p_1 \le L_1} \frac{1}{p_1} \Psi(x/p_1, p_1) \sum_{q^\alpha \le x/p_1,\, q<p_1} (\alpha - 1) q^{1-\alpha(1-\delta')}$$
$$\ll \sum_{z_1 < p_1 \le L_1} \frac{1}{p_1} \Psi(x/p_1, p_1) \sum_{q<p_1} q^{-1+2\delta'},$$

where $\delta' = (\log p_1)^{-1} \xi\big(\log(x/p_1)/\log p_1\big)$. By (2.2) and (2.3) we have

$$q^{2\delta'} \le \exp\left(2\xi\left(\frac{\log(x/p_1)}{\log p_1}\right)\right) \ll \log x \log_2 x,$$

for $z_1 < p_1 \le L_1$. Therefore using Lemma 1 we obtain:

$$W_4 \ll \log^2 x \sum_{z_1 < p_1 \le L_1} \frac{1}{p_1} \Psi(x/p_1, p_1) \ll R. \tag{3.4}$$

Similarly we have $W_6 \ll R$.

Now we come to the estimation of W_5 in (3.2). We consider separately the cases $p(m_1) < p_1$ and $p(m_1) = p_1$ and obtain

$$W_5 = \sum_{L_1 < p_1 \le L_2} \sum_{q^\alpha \le x/p_1,\, q<p_1} (\alpha - 1) q$$
$$\times \sum_{m_1 \le x/q^\alpha p_1,\, p(m_1)<p_1,\, (q,m_1)=1} \frac{1}{q + p_1 + \beta(m_1)} \tag{3.5}$$
$$+ O\left(\sum_{L_1 < p_1 \le L_2} \sum_{q^\alpha \le x/p_1,\, q<p_1} \frac{(\alpha-1)q}{p_1} \Psi(xq^{-\alpha} p_1^{-2}, p_1) \right).$$

Denoting by W_5' the main term on the right-hand side of (3.5) we may write

$$W_5' = \sum_{L_1 < p(m_1) < p_1,\, p(m_1)||m_1} + \sum_{L_1 < p(m_1) < p_1,\, p^2(m_1)||m_1} + \sum_{p(m_1) \le L_1}.$$

Then we have

$$W_5' = \sum_{L_1 < p_1 \le L_2} \sum_{L_1 < p_2 < p_1} \sum_{\substack{q^\alpha \le x/p_1 p_2 \\ q<p_1,\, q \ne p_2}} (\alpha - 1) q \sum_{\substack{m_2 \le x/q^\alpha p_1 p_2 \\ p(m_2)<p_2,\, (q,m_2)=1}} \frac{1}{q + p_1 + p_2 + \beta(m_2)}$$

$$+ O\bigg(\sum_{L_1 < p_1 \le L_2} \sum_{L_1 < p_2 < p_1} \sum_{q^\alpha \le x/p_1 p_2,\, q < p_1} (\alpha - 1) q p_1^{-1} \Psi(x/q^\alpha p_1 p_2^2, p_2) \bigg)$$
$$+ O\bigg(\sum_{L_1 < p_1 \le L_2} \sum_{p_2 \le L_1} \sum_{q^\alpha \le x/p_1 p_2,\, q < p_1} (\alpha - 1) q p_1^{-1} \Psi(x/q^\alpha p_1 p_2, p_2) \bigg).$$

Proceeding as before, we obtain

$$W_5 = W_5'' + O\bigg(\sum_{j=1}^{s} W_{7j} \bigg) + O\bigg(\sum_{j=2}^{s} W_{8j} \bigg) + O\bigg(\sum_{j=1}^{s-1} W_{9j} \bigg), \tag{3.6}$$

where
(3.7)
$$W_5'' = \sum_{p_1, \dots, p_s} \sum_{\substack{q^\alpha \le x/p_1 \dots p_s \\ q < p_1}} (\alpha - 1) q \sum_{\substack{m_s \le x/q^\alpha p_1 \dots p_s \\ p(m_s) < p_s,\, (q, m_s) = 1}} \frac{1}{q + p_1 + \cdots + p_s + \beta(m_s)},$$

where the ranges of summation in the above sums $p_1, \dots, p_s$ are $L_1 < p_1 \le L_2$, $L_1 < p_2 < p_1$, $\dots$, $L_1 < p_s < p_{s-1}$, and $s \le \log_3 x$ is a large number which will be chosen later and

$$W_{7j} = \sum_{p_1, \dots, p_j} \sum_{q^\alpha \le x/p_1 \dots p_j,\, q < p_1} \frac{(\alpha - 1) q}{p_1} \Psi(x/q^\alpha p_1 \dots p_{j-1} p_j^2, p_j), \tag{3.8}$$

$$W_{8j} = \sum_{p_1, \dots, p_{j-1}} \sum_{p_j \le L_1} \sum_{q^\alpha \le x/p_1 \dots p_j,\, q < p_1} \frac{(\alpha - 1) q}{p_1} \Psi(x/q^\alpha p_1 \dots p_j, p_j), \tag{3.9}$$

$$W_{9j} = \sum_{p_1, \dots, p_j} \sum_{q^\alpha \le x/p_1 \dots p_j,\, q < p_1} \frac{(\alpha - 1) q}{p_1} \Psi(x/q^{\alpha+1} p_1 \dots p_j, q). \tag{3.10}$$

Since

$$\frac{1}{q + p_1 + \cdots + p_s + \beta(m_s)} = \frac{1}{p_1 + \cdots + p_s} + O(q p_1^{-2}) + O(p_1^{-2} \beta(m_s)),$$

and

$$\sum_{\substack{m_s \le x/q^\alpha p_1 \dots p_s \\ p(m_s) < p_s,\, (q, m_s) = 1}} 1 = \Psi\bigg(\frac{x}{q^\alpha p_1 \dots p_s}, p_s \bigg) - \Psi\bigg(\frac{x}{q^{\alpha+1} p_1 \dots p_s}, p_s \bigg) + O(W_{7s}),$$

we have further

$$W_5'' = \sum_{p_1, \dots, p_s} \sum_{\substack{q^\alpha \le x/p_1 \dots p_s \\ q < p_1,\, \alpha \ge 2}} \frac{q}{p_1 + \cdots + p_s} \Psi(x/q^\alpha p_1 \dots p_s, p_s)$$

$$
\begin{aligned}
&+O\Bigg(\sum_{p_1,\dots,p_s}\ \sum_{q^\alpha\le x/p_1\dots p_s,\, q<p_1}\frac{(\alpha-1)q^2}{p_1^2}\Psi(x/q^\alpha p_1\dots p_s,p_s)\Bigg)\\
&+O\Bigg(\sum_{p_1,\dots,p_s}\ \sum_{q^\alpha\le x/p_1\dots p_s,\, q<p_1}\frac{(\alpha-1)q}{p_1^2}\sum_{\substack{m_s\le x/q^\alpha p_1\dots p_s\\ p(m_s)<p_s,\,(q,m_s)=1}}\beta(m_s)\Bigg)\\
&+O(W_{7s})=W_{10}+O(W_{11})+O(W_{12})+O(W_{7s}),\quad \text{say.}
\end{aligned}
$$

We estimate first W_{10}. We consider separately the cases $\alpha=2$ and $\alpha\ge 3$ and by using Lemmas 3 and 4 we obtain

$$
\begin{aligned}
W_{10}=&\sum_{p_1,\dots,p_s}\frac{\Psi(x/p_1\dots p_s,p_s)}{p_1+\dots+p_s}\sum_{q<p_1}q^{-1+2\delta_1}\big(1+O(\log_3 x/\log_2 x)\big)\\
&+O\Bigg(\sum_{p_1,\dots,p_s}\frac{\Psi(x/p_1\dots p_s,p_s)}{p_1+\dots+p_s}\sum_{\substack{q^\alpha\le x/p_1\dots p_s\\ q<p_1,\,\alpha\ge 3}}q^{1-\alpha(1-\delta_1)}\Bigg),
\end{aligned}
$$

where

$$
\delta_i=\frac{1}{\log p_s}\xi\left(\frac{\log(x/p_i\dots p_s)}{\log p_s}\right),\qquad i=1,2,\dots,s.
$$

Using partial summation and the prime number theorem we have

$$
\begin{aligned}
\sum_{q<p_1}q^{-1+2\delta_1}&=\int_{e^{1/\delta_1}}^{p_1}z^{-1+2\delta_1}\log^{-1}z\,dz\big(1+O(\log_3 x/\log_2 x)\big)\\
&=(\log_2 x)^{-1}p_1^{2\delta_1}\big(1+O(\log_3 x/\log_2 x)\big).
\end{aligned}
$$

Similarly

$$
\sum_{q^\alpha\le x/p_1\dots p_s,\, q<p_1,\,\alpha\ge 3}q^{1-\alpha(1-\delta_1)}\ll 1.
$$

Therefore we obtain

$$
W_{10}=\frac{1}{\log_2 x}\sum_{p_1,\dots,p_s}\frac{\Psi(x/p_1\dots p_s,p_s)}{p_1+\dots+p_s}p_1^{2\delta_1}\big(1+O(\log_3 x/\log_2 x)\big).
$$

By (4.13) of [**8**] we have $p^{\delta_i}=p^{\delta}(1+O(\log_3 x/\log_2 x))$ for $L_1<p\le L_2$, where $\delta=\delta_s$. Moreover by (4.6), (4.16), (4.18) and (4.31) of [**8**], we have

$$
\sum_{p_1,\dots,p_s}\frac{\Psi(x/p_1\dots p_s,p_s)}{p_1+\dots+p_s}=\left(D+O\left(\frac{\log_3^2 x}{\log_2 x}\right)\right)\sum_{L_1<p\le L_2}\frac{\Psi(x/p,p)}{p}.
$$

Similarly, in a way analogous to the above we have for W_{10}

$$
\begin{aligned}
W_{10}&=D(\log_2 x)^{-1}\big(1+O(\log_3^2 x/\log_2 x)\big)\sum_{L_1<p\le L_2}p^{-1+2\delta}\Psi(xp^{-1},p)\\
&=\frac12 D\log x\big(1+O(\log_3^2 x/\log_2 x)\big)\sum_{n\le x}\frac{1}{p(n)}.
\end{aligned}
\tag{3.12}
$$

Now we come to the estimation of W_{12} in (3.11). By the definition of $\beta(m)$ and Lemma 4 we have

$$W_{12} \leq \sum_{p_1,\dots,p_s} \sum_{q^\alpha \leq x/p_1\dots p_s,\, q<p_1} \frac{(\alpha-1)q}{p_1^2} \sum_{p<p_s} p\Psi(x/q^\alpha p_1\dots p_s p, p_s)$$
$$\ll \sum_{p_1,\dots,p_s} \sum_{p<p_s} \frac{p}{p_1^2} \Psi(x/p_1\dots p_s p, p_s) p_1^{2\delta_1} (\log_2 x)^{-1}.$$

By (4.19) of [**8**]

$$\sum_{p_1,\dots,p_s} \sum_{p<p_s} \frac{p}{p_1^2} \Psi(x/p_1\dots p_s p, p_s) \ll \frac{1}{\log_2 x} \sum_{L_1<p\leq L_2} \frac{\Psi(x/p,p)}{p} \ll R.$$

Similarly

$$W_{12} \ll R. \tag{3.13}$$

Similarly we have also

$$W_{11}, W_{7j}, W_{8j}, W_{9j} \ll R. \tag{3.14}$$

By putting (3.12)–(3.14) into (3.11) and (3.11), (3.14) into (3.6) and finally (3.3), (3.4) and (3.6) into (3.2) we get

$$W_1 = \frac{1}{2} D \log x \big(1 + O(\log_3^2 x/\log_2 x)\big) \sum_{n\leq x} \frac{1}{p(n)}.$$

Moreover, it is easy to prove that

$$W_2 \ll R,$$

which completes the proof of Theorem 1.

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