PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 46 (60), 1989, 7–12

VERTEX DEGREE SEQUENCES OF GRAPHS WITH SMALL NUMBER OF CIRCUITS

Ivan Gutman

Abstract. Necessary and sufficient conditions are determined for the numbers $p_1, p_2, \dots, p_n$ to be the vertex degrees of a connected graph with n vertices and cyclomatic number c, $c = 0, 1, 2, 3, 4, 5$.

Introduction

A partition $\boldsymbol{p} = (p_1, p_2, \dots, p_n)$ of the number $2m$ is said to be graphic if there exists a graph G with n vertices and m edges, such that the degree of the i-th vertex of G is equal to p_i, $i = 1, 2, \dots, n$. The characterization of graphic partitions and the study of graphs with prescribed degree sequences is a well elaborated part of graph theory [**1**,**2**].

Denote by $\mathbf{P}_N$ the set of all partitions of the integer N. If $\boldsymbol{a} \in \mathbf{P}_N$ then we say that $\boldsymbol{a}$ is of order N. Further, we present $\boldsymbol{a}$ as $(a_1, a_2, \dots, a_\alpha)$ and assume that $a_1 \geq a_2 \geq \cdots \geq a_\alpha > 0$. Of course, $a_1 + a_2 + \cdots + a_\alpha = N$.

If $\boldsymbol{a} \in \mathbf{P}_N$ then the conjugate partition of $\boldsymbol{a}$ is denoted by $\boldsymbol{a}^*$ and is defined as $\boldsymbol{a}^* = (a_1^*, a_2^*, \dots, a_{\alpha^*}^*)$ where $\alpha^* = a_1$ and $a_j^* = \max\{i \mid a_i \geq j\}$, $j = 1, 2, \dots, \alpha^*$. Then $\boldsymbol{a}^* \in \mathbf{P}_N$ and $a_1^* \geq a_2^* \geq \cdots \geq a_{\alpha^*}^* > 0$.

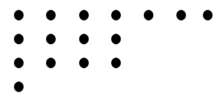
The partition $\boldsymbol{a}$ can be visualized by means of a Ferrers diagram [**1**] which is obtained by setting a_i dots in the i-th row, $i = 1, 2, \dots, \alpha$. This Ferrers diagram has then a_j^* dots in the j-th column, $j = 1, 2, \dots, \alpha^*$.

On Fig. 1 we present as an example the Ferrers diagram of the partition $(7, 4, 4, 1)$. It is immediately clear that the partition conjugate to $(7, 4, 4, 1)$ is $(4, 3, 3, 3, 1, 1, 1)$.

Definition. Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{P}_N$. If $\sum_{i=1}^r a_i \geq \sum_{i=1}^r b_i$ holds for all values of $r \in \mathbf{N}$, then we write $\boldsymbol{a}\, S\, \boldsymbol{b}$ and say that $\boldsymbol{a}$ is S-greater than $\boldsymbol{b}$.

If neither $\boldsymbol{a}\, S\, \boldsymbol{b}$ nor $\boldsymbol{b}\, S\, \boldsymbol{a}$, then the partitions $\boldsymbol{a}$ and $\boldsymbol{b}$ are said to be S-incomparable. S-incomparable partitions exist in $\mathbf{P}_N$, $N \geq 6$.

AMS Subject Classification (1980): Primary 05 C 75



(7, 4, 4, 1)

Fig. 1

If $\boldsymbol{a}$ is S-greater than $\boldsymbol{b}$, then the Ferrers diagram of $\boldsymbol{a}$ can be obtained from the Ferrers diagram of $\boldsymbol{b}$ by moving some dots upwards [**4**].

The relation S induces a partaial ordering of the set $\mathbf{P}_N$. Furthermore, $\langle \mathbf{P}_N; S \rangle$ is a lattice. This lattice has been introduced and examined by Snapper [**5**] and somewhat later by Ruch [**3**].

In [**4**] the following result has been proved.

Lemma 1. *If $\boldsymbol{a}$ is a graphic partition and $\boldsymbol{a}\, S\, \boldsymbol{b}$, then $\boldsymbol{b}$ is a graphic partition too.*

A proper consequence of Lemma 1 is that some graphic partitions are maximal with respect to the relation S. Maximal graphic partitions are necessarily mutually S-incomparable. Their structure is determined by the below lemma [**4**].

Let $\boldsymbol{a} = (a_1, a_2, \dots, a_\alpha)$ be a partition of order m, such that $a_1 > a_2 > \cdots > a_\alpha > 0$. Associate to $\boldsymbol{a}$ another partition $\boldsymbol{g} = \boldsymbol{g}[\boldsymbol{a}] = (g_1, g_2, \dots, g_n)$ via $g_j = a_j + j - 1$ and $g_j^* = a_j + 1$, $j = 1, 2, \dots, \alpha$. Note that $n = a_1 + 1$.

Lemma 2. *If $\boldsymbol{a}$ is a partition of the integer m into unequal parts, then $\boldsymbol{g}[\boldsymbol{a}] \in \mathbf{P}_{2m}$ and $\boldsymbol{g}[\boldsymbol{a}]$ is a maximal graphic partition. All maximal graphic partitions in $\mathbf{P}_{2m}$ are of the form $\boldsymbol{g}[\boldsymbol{a}]$.*

According to Lemma 2 the number of maximal graphic partitions of order $2m$ is equal to the number of partitions of m into unequal parts.

In Fig. 2 are presented the Ferrers diagrams of the six only possible maximal graphic partitions of order 16.

In [**4**] it has also been shown that the graph having a vertex degree sequence $\boldsymbol{g}[\boldsymbol{a}]$ is unique. This graph is connected and has cyclomatic number $c = m - a_1 = a_2 + \cdots + a_\alpha$. (Recall that the cyclomatic number of a connected graph with n vertices and m edges is equal to $m - n + 1$.)

Two auxiliary results

Lemma 3. *Let the partition $\boldsymbol{p} = (p_1, p_2, \dots, p_n)$ of order $2m$ be graphic. Then there exists a graph G with vertex degree sequence $\boldsymbol{p}$, such that* (a) *G is connected if $m \geq n - 1$,* (b) *G has $n - m$ components if $m \leq n - 1$.*

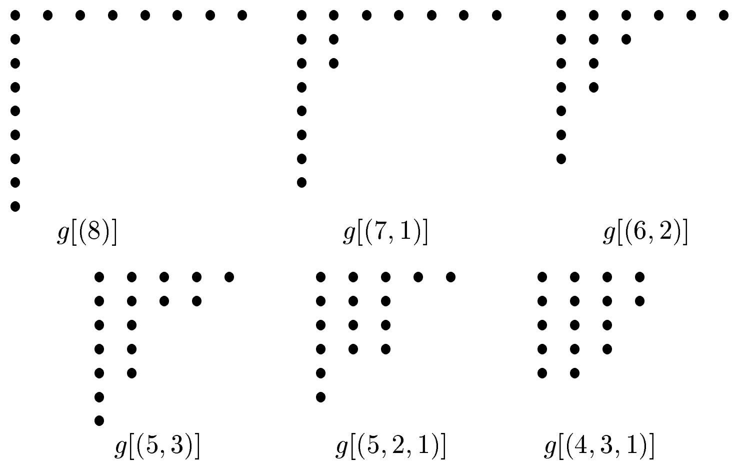
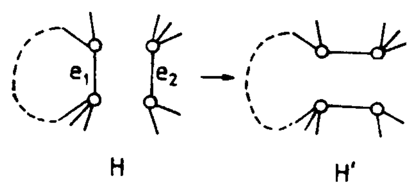


Fig. 2

Proof. Assume first that $m \geq n-1$ and H is a disconnected graph with degree sequence $\boldsymbol{p}$. Let e_1 and e_2 be edges belonging to two different components of H and let e_1 belong to a cycle. Such edges necessarily exist in a disconnected graph with $m \geq n-1$. Then the transformation $H \to H'$ will not change the degree sequence, but will decrease by one the number of components of H.



If H' is disconnected, we can repeat the procedure until a connected graph is obtained.

The proof for the case $m \leq n-1$ is analogous. □

LEMMA 4. *A connected graph with cyclomatic number c, $c > 0$, has at least m_c edges where*

(1) $$m_c = c + 1 + \left[\left(1+\sqrt{8c-7}\right)/2\right].$$

Proof is based on the observation that the graph with cyclomatic number c of the form

(2) $$c = x(x-1)/2 + y, \qquad x \in \mathbf{N}, \quad y \in \{1,2,\ldots,x\}$$

and the least number of edges is the graph $G(x,y)$ obtained by joining $y+1$ vertices of K_{x+1} to the (unique) vertex of K_1. Here K_n denotes the complete graph on n

vertices; note that $G(x,x) = K_{x+2}$. The number of edges of $G(x,y)$ is

$$m_c = x(x+1)/2 + y + 1, \qquad x \in \mathbf{N}, \quad y \in \{1, 2, \dots, x\}. \tag{3}$$

Eq. (1) is obtained from (2) and (3) by simple arithmetic reasoning. □

For the considerations which follow it is purposeful to extend the definition of the quantity m_c, eq. (1), by $m_c = 1$ for $c = 0$.

The main results

In this section we determine the conditions which a partition $\boldsymbol{p}$ of order $2m$ must satisfy in order to correspond to the degree sequence of a connected graph with given cyclomatic number c. Such a graph has $m - c + 1$ vertices and consequently its degree sequence $\boldsymbol{p}$ must have the property $p_1^* = m - c + 1$.

Denote by $\mathbf{P}_{2m}(c)$ the class of all partitions of the number $2m$ into exactly $m - c + 1$ parts. Hence, if $p \in \mathbf{P}_{2m}(c)$, then $p_1^* = m - c + 1$.

It is easy to verify that $\langle \mathbf{P}_{2m}(c); S \rangle$ is a lattice.

Bearing in mind Lemma 1, it is evident that if $\mathbf{P}_{2m}(c)$ contains graphic partitions, then some of them are maximal with respect to the relation $\mathbf{P}_{2m}(c)$. Denote by $\mathbf{P}_{2m}(c; \max)$ the set of maximal graphic partitions in $\mathbf{P}_{2m}(c)$.

From Lemma 2 it follows that the elements of $\mathbf{P}_{2m}(c; \max)$ are of the form $\boldsymbol{g}\big[(m-c, a_2, \dots, a_\alpha)\big]$, where $(a_2, \dots, a_\alpha)$ is a partition of the integer c into unequal parts, $c > 0$. (If $c = 0$ then the unique element of $\mathbf{P}_{2m}(c; \max)$ is $\boldsymbol{g}[(m)]$.)

Because of Lemma 3, if $m \geq m_c$ then every graphic partition in $\mathbf{P}_{2m}(c)$ is a vertex degree sequence of a connected graph.

The above observations can be summarized as follows.

Lemma 5. *A partition $\boldsymbol{p} \in \mathbf{P}_{2m}(c)$ is a vertex degree sequence of a connected graph with cyclomatic number c if and only if $\boldsymbol{q}\, S\, \boldsymbol{p}$ for some $\boldsymbol{q} \in \mathbf{P}_{2m}(c; \max)$. $\mathbf{P}_{2m}(c; \max)$ is non-empty if $m \geq m_c$.*

With these preparations we are able to prove the main results of the present paper, namely Theorems 1–5.

Theorem 1. *A partition $(p_1, p_2, \dots, p_n)$ of order $2m$, $m \geq 1$, is the vertex degree sequence of a tree iff $n = m + 1$.*

Theorem 2. *A partition $(p_1, p_2, \dots, p_n)$ of order $2m$ is the vertex degree sequence of a connected unicyclic graph iff $m \geq 3$, $n = m$, $p_1 \leq m - 1$ and $p_1 + p_2 \leq m + 1$.*

Theorem 3. *A partition $(p_1, p_2, \dots, p_n)$ of order $2m$ is the vertex degree sequence of a connected bicyclic graph iff $m \geq 5$, $n = m - 1$, $p_1 \leq m - 2$, $p_1 + p_2 \leq m + 1$ and $p_1 + p_2 + p_3 \leq m + 3$.*

THEOREM 4. *A partition* $(p_1, p_2, \dots, p_n)$ *of order* $2m$ *is the vertex degree sequence of a connected tricyclic graph iff* $m \geq 6$, $n = m - 2$, $p_1 \leq m - 3$ *and either* $(p_1 + p_2 \leq m + 1$ *and* $p_1 + p_2 + p_3 \leq m + 3)$ *or* $p_1 + p_2 \leq m$.

THEOREM 5. *A partition* $(p_1, p_2, \dots, p_n)$ *of order* $2m$ *is the vertex degree sequence of a connected tetracyclic graph iff* $m \geq 8$, $n = m - 3$, $p_1 \leq m - 4$ *and either* $(p_1 + p_2 \leq m$ *and* $p_1 + p_2 + p_3 \leq m + 3)$ *or* $p_1 + p_2 \leq m + 1$.

Proofs

The general strategy in proving Theorems 1–5 is the following. From Lemma 2 we know the number and the structure of the elements of $\mathbf{P}_{2m}(c; \max)$. Bearing in mind Lemma 5 we have just to find the condition(s) needed that an element of $\mathbf{P}_{2m}(c)$ is not S-greater than any element of $\mathbf{P}_{2m}(c; \max)$. In other words we have to avoid the partitions whose Ferrers diagrams are obtained by moving upwards a dot of the Ferrers diagram of $\boldsymbol{p} \in \mathbf{P}_{2m}(c; \max)$.

Denote by $[i, j]$ the dot in a Ferrers diagram lying in the i-th row and in the j-th column.

Proof of Theorem 1. Theorem 1 is a well known result [**2**]. We present its proof for reasons of completeness.

The unique element of $\mathbf{P}_{2m}(0; \max)$ is $\boldsymbol{g}[(m)] = (m, 1, 1, \dots, 1)$. In order to construct a partition which is S-greater than $(m, 1, 1, \dots, 1)$ we would have to move the dot $[m+1, 1]$ of the respective Ferrers diagram into position $[1, m+1]$. Such a transformation would, however, violate the condition $n = m + 1$.

Hence $(m, 1, 1, \dots, 1)$ is S-greater than any element of $\mathbf{P}_{2m}(0)$ i.e. every element of $\mathbf{P}_{2m}(0)$ is a vertex degree sequence of a tree. □

Proof of Theorem 2. The unique element of $\mathbf{P}_{2m}(1; \max)$ is $\boldsymbol{g}[(m-1, 1)] = (m-1, 2, 2, 1, 1, \dots, 1)$. A partition $\boldsymbol{p} \in \mathbf{P}_{2m}(1)$ will become S-greater than $\boldsymbol{g}[(m-1, 1)]$ if the dot $[3, 2]$ in the respective Ferrers diagram is moved either into the position $[1, m]$ or into the position $[2, 3]$ (c.f. Fig. 2). The transformation $[3, 2] \to [1, m]$ would increase p_1 by one. In order to avoid this we have to require $p_1 \leq m - 1$. The transformation $[3, 2] \to [2, 3]$ would increase p_2 by one, leaving p_1 unchanged. Bearing in mind the definition of the relation S we have to require $p_1 + p_2 \leq (m-1) + (2)$. This immediately yields Theorem 2. □

Proof of Theorem 3 is analogous thanks to the fact that $\mathbf{P}_{2m}(2; \max)$ also has a unique element $\boldsymbol{g}[(m-2, 2)] = (m-2, 3, 2, 2, 1, 1, \dots, 1)$.

Proof of Theorem 4. $\mathbf{P}_{2m}(3; \max)$ has two elements: $\boldsymbol{g}[(m-3, 3)]$ and $\boldsymbol{g}[(m-3, 2, 1)]$ (c.f. Fig. 2). In order to obtain a partition $\boldsymbol{p} \in \mathbf{P}_{2m}(3)$ which is S-greater than $\boldsymbol{g}[(m-3, 3)]$ we have to make one of the following three transformations:

(a) $[5, 1] \to [1, m-2]$ or $[2, 4] \to [1, m-2]$; (b) $[5, 1] \to [2, 5]$; (c) $[5, 1] \to [3, 3]$.

In order to avoid (a) we have to require $p_1 \leq m-3$. In order to avoid (b) we have to require $p_1+p_2 \leq (m-3)+(4)$. In order to avoid (c) we have to require $p_1+p_2+p_3 \leq (m-3)+(4)+(2)$.

In an analogous manner the conditions that $\boldsymbol{p} \in \mathbf{P}_{2m}(3)$ is not S-greater than $\boldsymbol{g}[(m-3,2,1)] = (m-3,3,3,3,1,1,\ldots,1)$ are $p_1 \leq m-3$ and $p_1+p_2 \leq (m-3)+(3)$.

All these conditions together result in Theorem 4. □

Proof of Theorem 5 is analogous to the proof of Theorem 4 since also $\mathbf{P}_{2m}(4;\max)$ has two elements.

From the above proofs it is evident that by continuing a similar way of reasoning and applying Lemmas 2 and 5 one can characterize the vertex degree sequences of connected graphs with cyclomatic numbers $c \geq 5$. These characterizations are somewhat more complicated because for $c \geq 5$, $|\mathbf{P}_{2m}(c;\max)| \geq 3$. A typical result of this kind is Theorem 6 which we state without proof.

THEOREM 6. *A partition* $(p_1,p_2,\ldots,p_n)$ *of order* $2m$ *is the vertex degree sequence of a connected pentacyclic graph iff* $m \geq 9$, $n = m-4$ *and either* (a) *or* (b) *or* (c) *holds*:

(a) $p_1+p_2 \leq m+1$.

(b) $p_1+p_2 \leq m$; $p_1+p_2+p_3 \leq m+3$; $p_1+p_2+p_3+p_4 \leq m+6$; $p_1+p_2+p_3+p_4+p_5 \leq m+8$.

(c) $p_1+p_2 \leq m-1$; $p_1+p_2+p_3 \leq m+3$; $p_1+p_2+p_3+p_4 \leq m+6$.

REFERENCES

[**1**] C. Berge, *Graphs and Hypergraphs*, North-Holland, Amsterdam, 1976, chapter 6.

[**2**] F. Harary, *Graph Theory*, Addison-Weslay, Reading, 1969, chapter 6.

[**3**] E. Ruch and I. Gutman, *The branching extent of graphs*, J. Combin. Infor. & System Sci. **4** (1979), 285–295.

[**4**] E. Snapper, *Group characters and nonnegative integral matrices*, J. Algebra **19** (1971), 520–535.

Prirodno-matematički fakultet
34001 Kragujevac, p.p. 60
Jugoslavija

(Received 18 11 1988)