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# A STRUCTURAL THEOREM FOR DISTRIBUTIONS HAVING S-ASYMPTOTIC

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**Abstract**. We prove that a distribution $T$ with an $S$-asymptotic related to $c(h)$ and to the cone $\Gamma$ has on the set $B+\Gamma$ a restriction which is a finite sum of derivatives of the functions $F_i$, continuous in $B+\Gamma$ and having some properties which imply that alle the $F_i(x+h)/c(h)$ converge uniformly for $x \in B$, when $h \in \Gamma$ and $\|h\| \to \infty$. If we know more about the distribution $T$ or about the cone $\Gamma$, then we can say more about the properties of $F_i$, $B$ is the ball $B(0,r)$.

**1. Introduction**. In the last few years many papers were published concerning the asymptotical behaviour of distributions at infinity. On of the reasons lies in the usefulness of these results in quantum field theory. $S$- asymptotic is one of the notions related to the asymptotical behaviour of distributions, introduced and elaborated in [**2**]. As $S$-asymptotic can be profitable in many studies and applications (see, for example, [**4**] and [**5**]), it is very useful to know the analytical expression of a distribution having $S$-asymptotic, especially if it is given by continous functions and their derivatives S. Plipović proved two theorems of this kind [**1**], but in the one dimensional case. Our theorem gives a different result and a different method of proof.

**2. Notations and definitions**. We denote by $\Gamma$ a cone in $R^n$ with the vertex at zero and $\sum(\Gamma)$ the set of real valued functions $c(h)$, $h \in \Gamma$, continuous and different from zero when $h \in \Gamma$. $B = B(0,r)$ will be the ball $\{x \in R^n, \|x\| < r\}$. The notations of the spaces of distributions are the same as in [**3**]; we use the $n$-dimensional case of distributions.

*Definition*. A distribution $T \in (D')$ has an $S$-asymptotic in the cone $\Gamma$ related to some $c(h) \in \sum(\Gamma)$ and with the limit $U \in (D')$ if the following limit exists

$$\lim_{h\in\Gamma,\|h\|\to\infty} \langle T(x+h)/c(h),\ \varphi(x)\rangle = \langle U, \varphi\rangle, \quad \varphi \in (D). \tag{1}$$

Then we write $T(x+h) \sim^s c(h) \cdot U(x),\ h \in \Gamma$.

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We know that if a filter $A$ has a countable base, then from the weak convergence in $(D')$, related to $A$, the strong convergence in $(D')$ follows as well. In this way relation (1) can be written in the form

$$\lim_{h\in\Gamma,\|h\|\to\infty} T(x+h)/c(h) = U \text{ in } (D'). \tag{2}$$

If $T \in (D')$ has an $S$-asymptotic related to $c(h)$, the set of distributions $Q \equiv \{T(x+h)/c(h),\ h\in\Gamma\}$ is weakly bounded in $(D')$. If this were not true, we would have a sequence $h_n \in \Gamma$ and $\varphi_0 \in (D)$ such that

$$|\ \langle T(x+h_n)/c(h_n),\ \varphi_0(x)\rangle - \langle U,\ \varphi_0(x)\rangle\ | \geq n, \qquad n \in N$$

which is not possible because of (1).

Now, from the weak boundedness of the set $Q$ it follows that $Q$ is bounded in $(D')$, as well.

**3. Structural theorem**. THEOREM. *If $T \in (D')$ has an $S$-asymptotic related to $c(h) \in \sum(\Gamma)$, then for the ball $B(0,r)$ there exist numerical functions $F_i, |\ i\ | \leq m$, continuous on $B(0,r)+\Gamma$, such that, for every $|\ i\ | \leq m$, $F_i(x+h)/c(h)$ converges uniformly for $x \in B(0,r)$ when $h \in \Gamma$, $\|h\| \to \infty$, and the restriction of the distribution $T$ on $B(0,r)+\Gamma$ can be given in the form $T = \sum_{|i|\leq m} D^i F_i$.*

For the proof of the theorem we need the following:

LEMMA. *If $T \in (D')$ and $T(x+h) \sim^s c(h)\cdot U(x)$, $h \in \Gamma$, then for a $B(0,r)$ and a $\Omega$ which is a relatively compact open neighbourhood of zero in $R^n$, there exists an $m \geq 0$ such that for every $\varphi,\psi \in (D^m_\Omega)$ the function $(T*\varphi*\psi)(x)$ is continuous for $x \in B(0,r)+\Gamma$ and the set of functions $\{(T_h*\varphi*\psi)(x),\ h\in\Gamma\}$ converges uniformly for $x \in B(0,r)$ to $(U*\varphi*\psi)(x)$, when*

$$h \in \Gamma,\ \|h\| \to \infty;\ \ T_h = T(x+h)/c(h)$$

*Proof*. Suppose that $T$ has an $S$-asymptotic related to $c(h)$; then the set $Q = \{T(+h)/c(h) \equiv T_h,\ h \in \Gamma\}$ is weakly bounded in $(D')$ and consequently bounded in $(D')$, as well. A necessary and sufficient condition that a set $B' \subset (D')$ is bounded in $(D')$ is: for every $\alpha \in (D)$ the set of functions $\{T*\alpha,\ T\in B'\}$ is bounded on every compact set $C$ belonging to $R^n$. Hence $\{T*\alpha,\ T\in B'\}$ defines a bounded set of regular distributions.

We denote by $\Omega$ an open neighbourhood of zero in $R^n$ which is relatively compact, $Cl(\Omega) = K$, $K$ is a compact set. For a fixed $\alpha$, supp $\alpha \subset K$, the linear mappings $\beta \to (T_h*\alpha)*\beta$ are continuous mappings of $(D_k)$ into $(E)$ because of the separate continuity of the convolution. As the set $\{T_h*\alpha,\ T_h\in Q\}$ is a bounded set in $(D')$, then for every ball $B(0,r)$ the set of mappings $\beta \to \{(T_h*\alpha)*\beta,\ T_h \in Q\}$ is the set of equicontinuous mappings of $(D_K)$ into $(L^\infty_B)$, $B = B(0,r)$. Now there exists a $m \geq 0$ such that the linear mappings $(\alpha,\beta) \to T_h*\alpha*\beta$ which map $(D_K)\times(D_K)$ into $(L^\alpha_B)$ can be extended to $(D^m_\Omega)\times(D^m_\Omega)$ in such a way that

$(\alpha,\beta)\to T_h*\alpha*\beta,\ T_h\in Q$, are equicontinuous mappings of $(D^m_\Omega)\times(D^m_\Omega)$ into $(L^\infty_B)$ (see for example the proof of Theorem XXII, p. 51 in [**3**]).

We saw that for every $\varphi,\psi\in(D^m_\Omega)$ and every $x\in B(0,r),\ h\in\Gamma$ the functions $(T_h*\varphi*\psi)(x)$ are continuos functions in $x$. From the relation $(T_h*\varphi*\psi)(x)=(T*\varphi*\psi)(x+h)/c(h)$ and from the properties of $c(h)$ it follows that $(T*\varphi*\psi)(y)$ is a continuous function for $y\in B(0,r)+\Gamma$ and $\varphi,\psi\in(D^m_\Omega)$.

It remains to prove that $T_h*\varphi*\psi$ converges to $U*\varphi*\psi$ in $(L^\infty_B)$ for $\varphi,\ \psi\in(D^m_\Omega)$. We know that $(D)$ is a dense subset of $(D^m),\ m\geq 0$. We can construct a subset $A$ of $(D_K)$ to be dense $(D^m_\Omega),\ Cl(\Omega)=K$. The set of functions $T_h*\alpha*\beta$ converges in $(L^\infty_B)$ for $\alpha,\ \beta\in A$, when $h\in\Gamma,\ \|h\|\to\infty$. Taking care of the equicontinuity of the mappings $(D^m_\Omega)\times(D^m_\Omega)$ into $(D^\infty_B)$, defined by $T_h*\varphi*\psi$, we can use the Banach-Stenhaus theorem to prove that $T_h*\varphi*\psi$ converges in $(L^\infty_B)$ when $h\in\Gamma,\ \|h\|\to\infty$.

*Proof of the Theorem.* We shall use, now relation (VI, 6; 23) from [**3**]

$$\Delta^{2k}*(\gamma E*\gamma E*T)-2\Delta^k*(\gamma E*\xi*T)+(\xi*\xi*T)=T \tag{4}$$

where $E$ is a solution of the iterated Laplace equation $\Delta^k E=\delta;\ \gamma,\ \xi\in(D_\Omega)$. We have only to choose the number $k$ large enough so that $\gamma E$ belongs to $(D^m_\Omega)$. Now, it is possible to take $F_1=\gamma E*\gamma E*T,\ F_2=\gamma E*\xi*T$ and $F_3=\xi*\xi*T$. All of these functions are of the form $F_i=T*\varphi_i*\psi_i,\ \varphi_i,\ \psi_i\in(D^m_\Omega),\ i=1,2,3$.

By the property of the convolution:

$$\begin{aligned}F_i(x+h)/c(h)=&(F_i(x)/c(h))*\tau_{-h}=(T*\varphi_i*\psi_i)*\tau_{-h}/c(h)\\=&((T*\tau_{-h})/c(h))*(\varphi_i*\psi_i)=T_h*\varphi_i*\psi_i.\end{aligned}$$

Hence, by the Lemma it follows that $F_i(x+h)/c(h)$ converges uniformly for $x\in B(0,r)$ when $h\in\Gamma,\ \|h\|\to\infty$.

**4. Consequences of the theorem 1**. It the functions $F_i,\ |\,i\,|\leq m$, have the property given in the theorem and if $\Gamma=R^n$, the regular distributions defined by the functions $F_i/c$ have an $S$-asymptotic related to $c_1(h)=1$. In the case $\Gamma=R^n$ the functions $F_i(x)/c(x)$ are continuous and $F_i(x)$ converge when $\|x\|\to\infty$ to the numbers $C_i$. Now,

$$\begin{aligned}\lim_{\|h|\to\infty}&\langle F_i(x+h)/c(x+h),\ \varphi(x)\rangle=\\&=\lim_{\|h|\to\infty}\int_{R^n}(F_i(x+h)/c(x+h))\ \varphi(x)dx,\qquad\varphi\in(D)\\&=\langle C_i,\varphi\rangle.\end{aligned}$$

**2**. If $\Gamma$ is a convex cone with nonempty interior, $\operatorname{int}\Gamma\neq\varnothing$, and if $T(x+h)\sim^s c(h)\cdot U(x),\ h\in\Gamma$, then for $x\in B(0,r)$

$$\lim_{h\in\Gamma,\|h|\to\infty}F_i(x+h)/c(h)=a_iU(x),$$

where $a_i$ are constants, $a_i \neq 0$.

*Proof*. If $\Gamma$ is a convex cone and $\int \Gamma \neq 0\varnothing$, then $U$ is of the form: $U(x) = b\exp(\ll x_0, x \gg)$, where $b \in R$, $x_0$ is a fixed element from $R^n$ and $\ll x,\ y \gg = \sum_{i=1}^n x_i y_i$ (see [**2**]).

All the functions $F_i$, $\mid i \mid \leq m$ are of the form $F_i = T * \varphi_i * \psi_i$, $\psi_i \in (D_\Omega^m)$. By the Lemma, for $x \in B(0,r)$, we have

$$\lim_{h\in\Gamma,\|h|\to\infty} F_i(x+h)/c(h) = \lim_{h\in\Gamma,\|h|\to\infty} (T(x+h)/c(x)) * \varphi_i * \psi_i = U * \varphi_i * \psi_i.$$

If $U(x) = b\exp(\ll x_0, x \gg)$, then $U * \varphi_i * \psi_i = a_i b\exp(\ll x_0, x \gg)$ which proves our assertion.

**3**. If $T$ belongs to a subspace $A'$ of $(D')$, then sometimes we can say more about $F_i$ from relation (3). As an illustration we shall examine the case $T \in (S')$.

In we suppose in our Theorem that $T \in (S') \subset (D')$, then we know not only that all the functions $F_i$, $\mid i \mid \leq m$, are continuous for $x \in B(0,r) + \Gamma$, but continuous functions of the slow growth. That means that $F_i(x) = (1+r^2)^q f_i(x)$, where $r = \|x\|$, $q \in R$ and $f_i$ are continuous and bounded functions.

*Proof*. If $T \in (S')$, then there exists a real number $q$ such that the set of distributions $\{T(x+h)/(1+\|h\|^2)^q,\ h \in R^n\} = W$ is bounded in $(D')$ (Theorem VI, p. 95 in [**3**]). We can repeat the first part of the Lemma's proof but with $c(h) = (1+\|h\|^2)^q$, $\Gamma = R^n$ and we shall obtain that there exists a $p \geq 0$ such that for $\varphi, \psi \in (D_\Omega^p)$ and $x \in R^n$ the function $(T * \varphi * \psi)(x)$ is continuous and $(T * \varphi * \psi)(x)/(1+\|x\|^2)^q$ is a bounded and continuous function. It remains only to choose the number $k$ in (4) large enough so that $\gamma E \in (D_\Omega^{\max(m,p)})$.

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