PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 45 (59), 1989, pp. 119-127

ASYMPTOTIC EXPANSIONS OF SCHWARTZ'S DISTRIBUTIONS

S. Pilipović

Abstract. We investigate the generalized asymptotic expansions of distributions and give some applications, mainly for the Weierstrass transform.

0. We give four definitions of the asymptotic expansion of distributions (for the third one see also [**1**] and [**10**]). Two of them are related to the shift operator and the other two are related to the dilation of a distribution. We give several structural assertions concerning these notions. In the last section we give applications of these notions, mainly for the Weirstrass transform. The example given in part 5 shows that for an ordinary function the generalized asymptotic expansion leads to a new classical Abelian result for its classical Weierstrass transform.

For the basic definitions of distribution spaces, see [**8**], and for the definition and properties of slowly varying functions at ∞. see [**9**]. Note that $\mathcal{D}'_+$ and $\mathcal{S}'_+$ are spaces of Schwartz distributions with elements having supports in $[0,\infty)$.

1. Denote by $c_m(k)$, $m \in \mathbf{N}$, a sequence of continuous positive functions defined on (a_m,∞), $a_m > 0$, such that

$$c_{m+1}(k) = o(c_m(k)),\ k \to \infty,\ (m \in \mathbf{N})$$

and by u_m, $m \in \mathbf{N}$, a sequence from $\mathcal{D}'$ such that $u_m \neq 0$, $m = 1,\dots,p$, $p < \infty$, $u_m = 0$, $m > p$, or $u_m \neq 0$, $m \in \mathbf{N}$. Denote by Λ the set of pairs of sequences $(c_m(k), u_m)$.

First, we reformulate Theorem and Corollary from [**3**]:

Proposition 1. *Let* $(c_m(k), u_m) \in \Lambda$ *and*

$$\lim_{k\to\infty} \langle (kx)/c_m(k),\ \varphi(x)\rangle = \langle g_m(x),\ \varphi(x)\rangle, \quad \varphi \in \mathcal{D}, \tag{1}$$

where $g_m \neq 0$ *if* $u_m \neq 0$, $m \in \mathbf{N}$. *Then for every* m *which* $u_m \neq 0$ *we have:*

AMS Subject Classification (1980): Primary 46 F 10, 46 F 12, 44 A 15 *(i)* $c_m(x) = x^{\nu m} L_m(x),\ x \in (a_m, \infty)$ *for some* $\nu_m \in \mathbf{R}$ *and some slowly varying function* L_m*;*

(ii) g_m *is a homogeneous distribution with the orther of homogeneity* ν*;*

(iii) $u_m \in \mathcal{S}'$

(iv) *if* $\nu \in \mathbf{R}\backslash\{-1,-2,\dots\}$ *then the limit in* (1) *exists in the sense of convergence in* $\mathcal{S}'$.

Remark 1. If we assume that $(c_m(k), u_m) \in \Lambda$ and $u_m \in \mathcal{D}'_+ (m = 1,\dots, p < \infty$ or $m \in \mathbf{N})$, then [**11**, §3. Theorem 3] implies that all the assertions in Proposition 1 hold without the restriction on ν in *(iv)* and with

$$g_m(x) = C_m f_{\nu m+1}(x),\ x \in \mathbf{R},\ C_m \neq 0,\ m = 1,\dots,p < \infty,\ \text{ or }\ m \in \mathbf{N}.$$

Recall that [**8**],

$$f_{\alpha+1}(t) = \begin{cases} H(t)t^\alpha/\Gamma(\alpha+1), & \alpha > -1 \\ D^n f_{\alpha+n+1}(t), & \alpha \leq -1,\ \alpha+n > -1 \end{cases} \qquad (t \in \mathbf{R}).$$

Denote by Λ_1 a subset of Λ such that $(c_m(k),\, u_m) \in \Lambda_1$ if (1) holds for all the m for which $u_m \neq 0$ and $g_m \neq 0$ $(m = 1,\dots,p < \infty$ or $m \in \mathbf{N})$, i. e. for which Proposition 1 holds.

Definition 1. Let $f \in \mathcal{D}'$ and $(c_m(k),\ u_m) \in \Lambda_1$ such that

$$\lim_{k\to\infty} \langle (((f(t) - \sum_{i=1}^{m} u_i(t))(kx))/c_m(k),\ \varphi(x)\rangle = 0, \quad \varphi \in \mathcal{D} \tag{2}$$

for $m = 1,\dots,p < \infty$ or $m \in \mathbf{N}$. Then we say that f has a quasiasymptotic expansion at ∞ of the first kind with respect to $(c_m(k), u+m)$ and we write

$$f \sim^{\text{q.e.}} \sum_{i=1}^{p(\infty)} u_i\ \ (c_m(k))\ \text{ at }\ \infty \tag{3}$$

Clearly, if (3) holds, then

$$f \sim^{\text{q.e.}} \sum_{i=1}^{p(\infty)} u'_i\ \ (k^{-1}c_m(k))\ \text{ at }\ \infty$$

Let $f \in \mathcal{D}'$ and (3) hold for some $(c_m(k),\ u_m) \in \Lambda_1$. Proposition 1 implies that for every m for which $u_m \neq 0,\ c_m(k) = k^{\nu m} L_m(k),\ (k > a_m)$.

With f satisfying Definition 1 we have:

PROPOSITION 2. *(i)* $f \in \mathcal{S}'$*; (ii) If* $\nu_m \notin -\mathbf{N}$*, then* (2) *exists in the sense of convergence in* $\mathcal{S}'$ *(for every* m *for which* $u_m \neq 0$*).*

Proof. Since $f(kx)/c_1(k) \to g_1 \neq 0$ in $\mathcal{D}'$, the Theorem from [**3**] mentioned implies that $f \in \mathcal{S}'$, whereas (ii) follows from Proposition 1 *(iv)*.

Remark 2. (Continuation of Remark 1). With the assumptions $f \in \mathcal{D}'_+$, $u_m \in \mathcal{D}'_+ (m = 1, \dots, p < \infty$ or $m \in \mathbf{N})$ Definition 1 generalizes the definition of the open quasiasymptotic expansion studied in [**11**, §10].

2. Another type of quasiasymptotic expansion at ∞ is given by the following definition:

Definition 2. Let $f \in \mathcal{D}'$ and $(c_m(k),\, u_m) \in \Lambda$. We write

$$f(kx) \sim^{\overline{\text{q.e.}}} \sum_{i=1}^{p(\infty)} u_i(x) \ \ (c_1(k)) \ \text{ at } \ \infty \tag{4}$$

iff for every $m \le p < \infty$ or $m \in \mathbf{N}$

$$\lim_{k\to\infty} \langle (f(kx) - \sum_{i=1}^{m} u_i(x) c_i(k))/c_m(k),\ \varphi(x)\rangle = 0,\ \ \varphi \in \mathcal{D} \tag{5}$$

In this case we say that f has a quasiasymptotic expansion at ∞ of the second type with respect to $(c_m(k),\, u_m)$.

Let us restrict this definition to a simpler case:

Definition 2'. Let $f \in \mathcal{S}'_+$ $(c_m(k),\, u_m) \in \Lambda$ with $u_m \in \mathcal{S}'_+ (m = 1, \dots, p < \infty$ or $m \in \mathbf{N})$ and let the limit (5) exist in the sense of convergence in $\mathcal{S}'$ (i. e. for $\varphi \in \mathcal{S}$). Then we say that f has a quasiasymptotic expansion at ∞ in $\mathcal{S}'_+$ of the second type with respect to $(c_m(k),\, u_m)$.

Proposition 3. *Let $(c_m(k), u_m) \in \Lambda$ and $f \in \mathcal{S}'_+$ satisfy the conditions of Definition 2'. Then:*

(i) $u_1(t) = A_1^1 f_{\alpha_1+1}(t)$, $t \in \mathbf{R}$, $c_1(k) = k^{\alpha_1} L_1(k)$, $k > a_1$, for some $a_1 > 0$, and some L_1, where $A_1^1 \neq 0$;

(ii) for $m = 2, \dots, p < \infty$ (if $p \ge 2$) or $m \in \mathbf{N}$, $m \ge 2, u_m$ is the solution of a differential equation of the form

$$l_{\alpha m-1}(\cdots(l_{\alpha_1}(u_m))\cdots) = A_m f_{\alpha m+1} (A_m \in \mathbf{R}), \tag{6}$$

where $l_\nu(u) \equiv xu' - \nu u (\nu \in \mathbf{R},\, u \in \mathcal{D}')$.

If in (6) $A_m \neq 0$, then $c_m(k) = k^{\alpha m} L_m(k)$, $k > a_m$, for some $a_m > 0$ and some L_m.

(iii) if $\alpha_1 > \alpha_2 > \cdots > \alpha_p$ (for $p < \infty$) or $\alpha_i < \alpha_j$ for $i > j$, $j \in \mathbf{N}$, then for $m = 2, \dots, p < \infty$ or $m \in \mathbf{N}$, $m \ge 2$,

$$u_m = \sum_{j=1}^{m-1} A_i^m f_{\alpha j+1} + \Big(A_m / \sum_{i=1}^{m-1} (\alpha_m - \alpha_i)\Big) f_{\alpha m+1}, \tag{7}$$

where A_j^m, $j = 1, \dots, m-1$ are suitable constants.

Proof . *(i)* For $m = 1$ we have:

$$\lim_{k\to\infty} \langle f(kx)/c_1(k),\ \psi(x)\rangle = \langle u_1,\ \varphi\rangle,\, \varphi \in \mathcal{S}.$$

The well-known assertion [**11**], §3, Theorem 1] implies *(i)*.

(ii) First note that $f_{\alpha+1}$ satisfies the differential equation $l_\alpha(u)=0$.

For $m=2$ we have

$$\lim_{k\to\infty}\rangle(f(kx)-u_1(x)c_1(k))/c_2(k),\ \varphi(x)\rangle=\langle u_2,\varphi\rangle\in\mathcal{S}. \tag{8}$$

This implies

$$\begin{aligned}\lim_{k\to\infty}&\langle(xkf'(kx)-c_1(k)xu_1'(x))/c_2(k),\ \ \varphi(x)\rangle\\ &=\langle xu_2'(x),\ \psi(x)\rangle,\quad \varphi\in\mathcal{S}.\end{aligned}\tag{9}$$

Thus, multiptying (8) with $-\alpha_1$ and adding that to (9) we obtain

$$\lim_{k\to\infty}\langle(l_{\alpha_1}f)(kx)/c_2(k),\ \varphi(x)\rangle=\langle(l_{\alpha_1}u_2)(x),\ \varphi(x)\rangle,\qquad \varphi\in\mathcal{S}.$$

If $l_{\alpha_1}u_2\neq 0$, then by the same arguments as for $m=1$ we obtain

$$\begin{gathered}c_2(k)=k^{\alpha_2}L_2(k)\ \ (k>a_2)\ \text{ for some }\ \alpha_2,\\ l_{\alpha_1}u_2=A_2f_{\alpha_2+1}\ \text{ for some }\ A_2\neq 0.\end{gathered}$$

Instead of finishing this part of the proof by induction, we give the proof for $m=3$. After that the proof by induction becomes trivial. We have

$$\begin{aligned}&\lim_{k\to\infty}\langle((l_{\alpha_1}f)(kx)-(l_{\alpha_1}u_2)(x)c_2(k))/c_3(k),\ \ \varphi(x)\rangle\\ &=\langle(l_{\alpha_1}(u_3)(x),\ \psi(x)\rangle,\qquad \varphi\in\mathcal{S}.\end{aligned}$$

Since $l_{\alpha_1}(u_2)=A_2f_{\alpha_2+1}$ we have

$$\begin{aligned}&\lim_{k\to\infty}\langle((l_{\alpha_2}(l_{\alpha_1}f))(kx)/c_3(k),\ \ \varphi(x)\rangle=\\ &=\langle(l_{\alpha_2}(l_{\alpha_1}(u_3))(x),\ \varphi(x)\rangle,\qquad \varphi\in\mathcal{S}.\end{aligned}$$

Now, as in the case $m=2$ we derive the necessary conclusions.

(iii) The particular solution of (5) is given by the last member in (6) because of the identity $l_\nu(f_{\mu+1})=(\mu-\nu)f_{\mu+1}$, where μ,ν are arbitrary elements from $\mathbf{R}$.

Note that $l_\nu(l_\mu u)=l_\mu(l_\nu u),\ (u\in\mathcal{D}')$.

Remark 3. If for some $m,\alpha_m=\alpha_{m-1}$ and $A_m\neq 0$, then equation (5) does not have such a "nice" solution. If we assume that the sequence $c_m(k)$ satisfies the stronger condition:

$$c_{m+1}(k)/c_m(k)=O(k^{-\varepsilon_m}),\ \varepsilon_m>0,\ m=1,\dots,p<\infty\ \text{ or }\ m\in\mathbf{N},$$

then by using the properties of slowly varying functions one can deduce that

$$\alpha_1>\alpha_2\ldots>\alpha_p\ \text{ or }\ \alpha_i<\alpha_j,\ i>j,\ i,\ j\in\mathbf{N}.$$

Remark 4. Let us assume that Definition 2 holds for f and $(c_m(k), u_m)$ and that $f \in \mathcal{S}'_+$ and $u_m \in \mathcal{S}'_+$. The question is whether the limit in (5) can be extended to the whole of $\mathcal{S}$? Note that Proposition 3 does not give an answer to this question.

Remark 5. If we assume that the assumptions of Remark 1 are satisfied for f and $(c_m(k), u_m)$, and if $c_m(k)$ are polynomials, then the quasiasymptotic expansion of the first type is equivalent to the quasiasymptotic expansion of the second type. In general this does not hold. For example, we have:

$$x^5 \ln \mid x \mid + x^4 \sim^{\text{q.e.}} x^5 \ln \mid x \mid + x^4 \ (c_1(k) = k^4 \ln k,\ c_2(k) = k^4,\ k > 1),$$

(see Definition 1) and

$$(kx)^5 \ln(k \mid x \mid) + (kx)^4 \sim^{\overline{\text{q.e.}}} (x^5 \ln \mid x \mid) k^5 \ln k + (x^5 \ln \mid x \mid) k^5 + x^4 k^4$$

(see Definition 2).

This example shows that Definition 1 is more natural than Definition 2 (or 2').

3. Let d_m be a sequence of positive continuous functions different from zero in (a_m, ∞), $a_m > 0$, and $d_{m+1}(h) = o(d_m(h))$, $h \to \infty (m \in \mathbf{N})$. Denote by u_m a sequence from $\mathcal{D}'$ such that $u_m \neq 0, m = 1, \dots, p < \infty\, u_m = 0, m > p$, or $u_m \neq 0$, $m \in \mathbf{N}$. We denote by $\sum$ the set pairs of sequences $(d_m(h), u_m)$.

The following definition is a slight modification of a definition from [**10**]. We adapt it in the sense of the notation given above.

Definition 3. An $f \in \mathcal{D}'$ has an S-asymptotic expansion of second type with respect to $(d_m(h), u_m) \in \sum$ if

$$\lim_{h \to \infty} \langle (f(x+h) - \sum_{i=1}^{m} u_i(x) d_i(h))/d_m(h),\ \varphi(x) \rangle, \quad \varphi \in \mathcal{D}, \tag{10}$$

for $m = 1, \dots, p < \infty$ or $m \in \mathbf{N}$.

In this case we write

$$f(x+h) \sim^{\overline{s.e.}} \sum_{i=1}^{p(\infty)} u_i(x) d_i(h).$$

Remark 6. Definition 3 is a generalization of a corresponding definition for the space $\mathcal{S}'$ given in [**1**].

PROPOSITION 4. *Let $f \in \mathcal{D}$ and $(d_m(h), u_m) \in \sum$ satisfy the condition of Definition 3. Then we have:*

(i) $u_1(t) = A_1^1 \exp(\alpha_1 t)$, $t \in \mathbf{R}$, $A_1^1 \neq 0$, $\alpha_1 \in \mathbf{R}$, $d_1(h) = \exp(\alpha_1 h) L_1$ $(\exp h)$, $h > a$ (for some a_1 and some L_1);

(ii) for $m = 2$, $p < \infty$ (if $p \geq 2$) or $m \in \mathbf{N}$, u_m is the solution of the equation

$$L_{\alpha_m - 1}(\cdots (L_{\alpha_1} u_m) \dots) = A_m \exp(\alpha_m t),\ A_m \in \mathbf{R}, \tag{11}$$

where $L_\nu u = u' - \nu u (u \in \mathcal{D}',\ \nu \in \mathbf{R})$.

If in (11) $A_m \neq 0$, *then* $d_m(h) = \exp(\alpha_m h) L_m(\exp\ h),\ \ h > a_m$;

(iii) for $m = 1, \dots, p < \infty$ *or* $m \in \mathbf{N}$,

$$u_m(t) = \sum_{i=1}^{m-1} A_i^m \exp(\alpha_i t) + p_{m-1}(t) \exp(\alpha_m t),\ (t \in \mathbf{R}) \tag{12}$$

where $A_i^m,\ i = 1, \dots, m-1$, *are suitable constants and* p_{m-1} *is a suitable polynomial of degree* $\leq m - 1$.

Proof. *(i)* This is a direct consequence of [**6**, Theorem 5].

(ii) We have

$$L_\alpha(\exp\ \beta t) = (\beta - \alpha) \exp(\beta t),\ \ t \in \mathbf{R},\ \ (\alpha, \beta \in \mathbf{R})$$
$$L_\alpha(L_\beta u) = L_\beta(L_\alpha u),\ (u \in \mathcal{D}').$$

Let $m = 2$. We have:

$$\lim_{h\to\infty} \langle (f(x+h) - u_1(x) d_1(h))/d_2(h),\ \varphi(x)\rangle = \langle u_2(x),\ \varphi(x)\rangle,\ \ \varphi \in \mathcal{D}, \tag{13}$$

$$\lim_{h\to\infty} \langle (f'(x+h) - u_1'(x) d_1(h))/d_2(h),\ \varphi(x)\rangle = \langle u_2'(x),\ \varphi(x)\rangle,\ \ \varphi \in \mathcal{D}, \tag{14}$$

Multiplying (13) by —α_1 and adding that to (14) we obtain

$$\lim_{h\to\infty} \langle (L_{\alpha_1} f)(x+h)/d_2(h),\ \ \varphi(x)\rangle = \langle (L_{\alpha_1} u_2),\ \ \varphi(x)\rangle,\ \ \varphi \in \mathcal{D}.$$

As in *(i)* from [**7**, Theorem 5] the assertion for $m = 2$ follows. Then by induction we complete the proof of *(ii)*.

(iii) The proof follows from the fact that, for a suitable polynomial p_{m-1} of order $\leq m - 1,\ p_{m-1}(t) \exp(\alpha_m t),\ \ t \in \mathbf{R}$, is the patricular solution of (11).

4. Denote by $\sum_1$ a subset of $\sum$ consisting of elements $(d_m(h),\ u_m)$ for which we have

$$\lim_{h\to\infty} \langle (u_m(x+h)/d_m(h),\ \varphi(x)\rangle = \langle g_m(x),\ \varphi(x)\rangle, \\ \varphi \in \mathcal{D},\ g_m \neq 0,\ m = 1, \dots, p < \infty \text{ or } m \in \mathbf{N}. \tag{15}$$

As remarked above, from [**7**, Theorem 5] it follows that

$$d_m(h = \exp(\alpha_m h) L_m(\exp\ h) \neq 0,\ \ h > a_m,\ \alpha_m \in \mathbf{R}, \tag{16}$$

where L_m is a suitable slowly varying function, $m = 1, \dots, p < \infty$ or $m \in \mathbf{N}$, and

$$g_m(x) = C_m \exp(\alpha_m x),\ C_m \neq 0,\ m = 1, \dots, p < \infty \text{ or } m \in \mathbf{N}.$$

So, we have that the first component of an element from $\sum_1$ is a sequence for which (16) holds $m = 1, \dots,\ p < \infty$ or $m \in \mathbf{N}$.

Definition 4. Let $f \in \mathcal{D}'$ and $(d_m(h),\ u_m) \in \sum_1$. If

$$\lim_{h\to\infty} \langle (f(x+h) - \sum_{i=1}^{m} u_1(x+h))/d_m(h),\ \varphi(x)\rangle = 0, \tag{17}$$
$$\varphi \in \mathcal{D},\ \ m = 1, \dots, p < \infty \text{ or } m \in \mathbf{N},$$

then we say that f has an S-asymptotic expansion at ∞ of the first kind with respect to $(d_m(h),\ u_m)$, and we write

$$f(x) \sim^{\text{s.e.}} \sum_{m=1}^{p(\infty)} u_m(x) \qquad (d_m(h)). \tag{18}$$

Clearly, if (18) holds, then

$$f'(x) \sim^{\text{s.e.}} \sum_{m=1}^{p(\infty)} u'_m(x) \qquad (d_m(h)).$$

Recall the definition of the space $\mathcal{K}'_1$, introduced by Hasumi:

$$\mathcal{K}_1 = \{\varphi \in C^\infty;\ \sup_{i<m\ x\in\mathbf{R}} \{ch(mx) \mid \varphi^{(i)}(x) \mid < \infty\},\ \ m = 0, 1, \dots\};$$

$\mathcal{K}'_1$ is its dual. Denote by $\mathcal{K}'_{1,ar}$ the space of all $f \in L^1_{1os}$ such that $f\varphi \in L^1$ for every $\varphi \in \mathcal{K}_1$ (see [**4**]).

PROPOSITION 5. *Let f satisfy (18).*

(i) Assume $f \in \mathcal{K}'_1$, $u_m \in {}'_1$ and let the slowly varying functions L_m in (16) be monotonous (for sufficiently large arguments) $m = 1, \dots, p < \infty$ or $m \in \mathbf{N}$. Then the limit in (17) *may be extended from $\mathcal{D}$ to $\mathcal{K}_1$.*

(ii) Asume, $f \in \mathcal{S}'$, $u_m \in \mathcal{S}'$ and $d_m(h) = h^{\alpha m} L_m(h)$, where every L_m is monotonous, $m = 1, \dots, p < \infty$ or $m \in \mathbf{N}$. Then the limit (17) *may be extended from $\mathcal{D}$ to $\mathcal{S}$.*

Proof. From [**6**] it easily follows that with the given assumptotions, (16) can be extended from $\mathcal{D}$ to $\mathcal{K}_1$, i.e., $\mathcal{S}$. This implies the assertions.

The ordinary asymptotic expansion implies this type of distributional asymptotic expansion. Namely, we have:

PROPOSITION 6. *Let $f \in \mathcal{K}'_{1,ar}$, and let u_m and $d_m, m = 1, \dots, p < \infty$ or $p \in \mathbf{N}$, satisfy the assumptions of Proposition 5 (i).*

If

$$f(x) \sim \sum_{m=1}^{p(\infty)} u_m(x) \ \ \textit{as}\ \ x \to \infty \ \ \textit{(in the ordinary sense)},$$

then

$$f(x) \sim^{\text{s.e.}} \sum_{m=1}^{p(\infty)} u_m(x), \quad (d_m(h) = \exp(\alpha_m h) L_m(\exp\ h)) \\ m = 1, \dots, p \text{ or } m \in \mathbf{N}.$$

The proof of this proposition is similar to the proof of Proposition 3 in [**4**], so we shall omit it.

Let us give two examples. If

$$f(x) = \sqrt{x^2 + x},\ x > 0 \text{ and } f(x) = 0,\ x \le o,$$

then we have

$$f(x) \sim^{\text{s.e.}} x + \binom{1/2}{1} + \binom{1/2}{2}(1/x)_+ + \cdots \binom{1/2}{n}(1/x)_+^{n-1} + \cdots (d_m), \tag{19}$$

where $d_m(h) = h^{2-m},\ h > 0, m \in \mathbf{N}$. Formula (19) quite naturaly follows from the ordinary asymptotic expansion of f at ∞. Note that the S-asymptotic expansion of f of the second type at ∞ is much more complicate, and is not equal to (19).

In the example which follows we construct a function which has an S-asymptotic expansion of the first type but has no ordinary asymptotic expansion.

Let $\psi(t) = 1,\ t \in (n - 2^{-n}, n + 2^{-n}),\ n \in \mathbf{N}$, and $\psi(t) = 0$ outside of these intervals. Let $\psi_\alpha(x) = e^{\alpha x} \int_0^x \psi(t)dt,\ x \in \mathbf{R},\ \alpha \in \mathbf{R}$. Since $\int_0^x \psi(t)dt \to 2$ as $x \to \infty$, we have that $\psi_\alpha(x) \sim 2e^{\alpha x},\ x \to \infty$ but $\psi'_\alpha(x)$ does not have an ordinary asymptotic behaviour [**4**]. Let (α_j) be a strictly decreasing sequence of positive numbers $\theta \in C^\infty \equiv 1$ for $x > 1, \theta \equiv 0$ for $x < 1/2$ and let $f(x) = \sum_{m=1}^{\infty} \varphi_{\alpha i}(x)\theta(x - i),\ x \in \mathbf{R}$. We have

$$f(x) \sim \sum_{i=1}^{\infty} \varphi_{\alpha i}(x), \quad x \to \infty$$

with respect to the sequence $\{2e^{\alpha i x}; i \in \mathbf{N}\}$. This implies that

$$f(x) \sim^{\text{s.e.}} \sum_{m=1}^{\infty} \varphi_{\alpha i}(x) \text{ with respect to } \{2e^{\alpha i x},\ i \in \mathbf{N}\}$$

and

$$F(x) = f'(x) \sim^{\text{s.e.}} \sum_{i=1}^{\infty} \varphi'_{\alpha i}(x) \text{ with respect to } \{2e^{\alpha i x},\ i \in \mathbf{N}\} \tag{20}$$

but $F(x)$ does not have an ordinary asymptotic expansion.

5. In this part we shall give some applications. First, we note that for the distributional Laplace trasform the quasiasymptotic expansion of the first type of an orginal at ∞ implies the ordinary asymptotic expansion of its Laplace transform at

0. This is studied in [**11, 12**] and in a forthcoming paper of the author. Similary, for the distributional Stieltjes transform we apply this notion, in a separate forthcoming paper, for obtaining the corresponding Abelain type results at ∞.

We shall give in this section some applications of the S-asymptotic expansivn of the first kind. As a direct consequence of [**5**] we have:

PROPOSITION 6. *Let*

$$f(x) \sim^{\text{s.e.}} \sum_{i=1}^{p} u_i(x) \ \ (d_m(h)),\ p < \infty.$$

Then

$$f(x) = \sum_{i=1}^{p} u_i(x) \ x > A \ \textit{for some} \ A$$

iff for every positive continuous function $d(h),\ h > A$

$$\lim_{h\to\infty} \langle (f - \sum_{i=1}^{p} u_m)(x+h)/d(h),\ \varphi(x)\rangle = 0,\ \varphi \in \mathcal{D}.$$

The Weiestrass kernel is defined by

$$k(s,t) = (4\pi t)^{-1/2} e^{-s^2/(4t)},\ s \in C,\ t \in (0,1).$$

Obviously, for any $s \in C$ and $t \in (0,1)$, $k(s-x,t) \in \mathcal{K}_1$. The Weiestrass transform of an $f \in \mathcal{K}_1'$ is defined by $(W_t f)(s) = \langle f(x),\ k(s-x,t)\rangle$ [**4**]. From [**4**, Proposition 4] the following proposition follows directly:

PROPOSITION 7. *Assume that the assumptions of Proposition 5 (i) are satisfied. Then (in the ordinary sense) for any* $s \in C$

$$W_t f(s+h) \sim \sum_{m=1}^{p(\infty)} A_{s,t,m} d_i(h),\ h \to \infty,$$

where $A_{s,t,m} = (W_t u_m)(s),\ m = 1, \ldots p < \infty$ or $m \in \mathbf{N}$.

Remark 7. Example (20) and this proposition show that for the classical Weiestrass transform the notion of S-asymptotic expansion implies new classical results for the behaviour of its transform.

REFERENCES

[1] Ю. А. Бричков, *Асимптотические разложенийобобщенных функций*, И ТМФ **5** (1970), 98–109.

[2] I. M. Gel'fand, G. E. Shilov, *Generalized Functions, Vol.* 1, *Properties and operations*, Acad. Press, New York and London, 1964.

[3] S. Pilipović, *On the quasiasymptotic behaviour of Schwartz distributions*, Math. Nachr., **137** (1988), 19–25.

[4] S. Pilipović, *Asymptotic behaviour of the distributional Wierstrass transform*, Appl. Anal., **25** (1987) 171–179.

[5] S. Pilipović, *Remarks on the supports of distributions*, Glasnik Mat., **22**(**42**)(1987), 375–380.

[6] S. Pilipović, *On the S-asymptotic of tempered and K-distributions, Part I and II*, Rev. Res. Sci. Math. Univ. Novi Sad **15** (1989), 47–58, 59–67.

[7] S. Pilipović, B. Stanković, *On the S-asymptotic of a distribution*, Pliska, (to appear).

[8] L. Schwartz, *Theorie des distributions Vols I, II, Hermann*, Paris, 1957–1999.

[9] E. Seneta, *Regularly varying functions*, Springer Verlag, Berlin-Haidelberg-New York, 1976

[10] B. Stanković, *S-asymptotic expansion of distributions*, Internat. J. Math. Sci. (to appear).

[11] В. С. Владимиров, Ю. Н. Дрожжинов, Б. И. Завьялов, *Моогомерные Тауберов теоремы для обобщенных функций*, Наука, Москва, 1986.

[12] Б. И. Завьялов, *Автомодельная асимптитика глекромагнимных форм-факторов и поведение их фурь-образов в окрестности светности светового конуса*, ТМФ **17** (1973), 178–188.

Institut za matematiku
Univerzitet u Novom Sadu
Ilije Đ urića 4
21000 Novi Sad

(Received 08 03 1988)