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ON LINEAR TOPOLOGICAL RIESZ SPACES WITHOUT CONVEXITY CONDITIONS

Stojan Radenović

Abstract. We consider whether the space associated with an l.t.R.s. (E, C, t) is l.t.R.s. We have shown that any l-ideal in an ultra-DF (resp. countably quasibarrelled, locally topological, ultra-b-barrelled, ultra D_b) Riesz space is space of the same type with respect to the relative topology.

Throughout this paper (E, C, t) will denote a separated linear topological Riesz space (l.t.R.s.) over the field of real or complex numbers. The notions concerning the theory of l.t.s.'s. (resp. l.t.R.s.'s.) without convexity conditions can be found in [**1**] (resp [**3**] and [**9**]). We give here only the basic ones.

A string in (E, t) is a sequence $(V_n)_{n\in N}$ of subsets of E which are circled, absorbing and satisfy $V_{n+1} + V_{n+1} \subset V_n (n = 1, 2, 3, \dots)$. The strung $(V_n)_{n\in N}$ is said to be topological (resp. closed, bornivorous) if each V_n is a t-neighbourhood of 0 in E (resp. t-closed, bornivorous). It is clear that each circled t-neighbourhood of 0 in E generates a (non-uniquely determined) topological string. A string (resp. closed string) $(V_n)_{n\in N}$ in an l.t.s. (E, t) is called locally topological (resp. closed locally topological) if $V_n \cap B$ is a neighbourhood of 0 for the topology induced by t on B for all $n \in \mathbf{N}$ and all t-bounded balanced sets B.

A function $p : E \to R$ satisfying the conditions:

(a) $p(x) \geq 0$ for each $x \in E$,

(b) $p(x + y) \leq p(x) + p(y)$ for each $x, y \in E$,

(c) $p(\lambda x) \leq p(x)$ for each $x \in E$ and each $\lambda \in K$, $\mid \lambda \mid \leq 1$,

(d) if $\lambda_n \in K, \lambda_n \to 0$ and $x \in E$, then $p(\lambda_n x) \to 0$,

is called an (F)-seminorm. If, moreover, $p(x) = 0$ implies $x = 0$, p is called an (F)-norm. (F)-seminorms in a certain sense have a similar role in the theory of l.t.s.'s as seminorms have in the theory of locally convex spaces (l.c.s.). Namely,

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each linear topology on a vector space can be determined by a family of continuous (F)-seminorms.

It is known that to each topological string in an l.t.s. corresponds a continuous (F)-seminorm and vice versa [**1**]. A σ-barrel (resp. bornivorous σ-barrel) in an $l.t.s.(E,t)$ is a closed string (resp. closed bornivorous string) $\mathcal{V} = (V^{(j)})_{j\in N}$ which is a countable intersection of closed topological strings, i.e. $V^{(j)} = \cap_{n=1}^{\infty} V_n^{(j)}$, where $\mathcal{V}_n = (V_n^{(j)})_{j\in N}$ are closed topological strings for each $n \in \mathbf{N}$. An l.t.s. (E,t) is called countable barrelled (resp. countably quasibarrelled) if every σ-barrel (resp. bornivorous σ-barrel) in it is topological. An l.t.s. (E,t) is an ultra-DF space, if it is countable quasibarrelled and if it has a fundamental sequence of bounded sets. An l.t.s. (E,t) is called locally topological (resp. ultra-b-barrelled) if each locally topological (resp. closed locally topological) string in it is topological [**1, 2**]. An l.t.s. (E,t) is called ultra-D_b if it is ultra-b-barrelled with a fundamental sequence of bounded sets [**2**].

The string $(V_n)_{n\in N}$ in an l.t.R.s. (E,C,t) is said to be solid if each V_n is a solid subset in the Riesz space (E,C).

The rest of our terminology is taken from [**1**] or [**9**]. In particular, when we say, for example "barrelled space", that means "barrelled in the category of l. t. s.'s".

Definition 1. An (F)-seminorm an a Riesz space (E,C) is called a Riesz (F)-seminorm if from $\mid x \mid \leq \mid y \mid$ with x and y in E it follows that $p(x) \leq p(y)$.

From the following proposition it follows that Riesz (F)-seminorms have the same role in the theory of l.t.R.s. as Riesz seminorms have in the theory of locally convex Riesz spaces [**9**, Theorem (6.3)].

PROPOSITION 1. *Let (E,C,t) be an ordered linear topological space. Then (E,C,t) is a l.t.R.s. if and only if t is determined by a family of continous Riesz (F)- seminorms on E.*

Proof. Let (E,C,t) be a l.t.R.s. Then there exists a solid t-neighbourhood V of 0 which generates a solid topological string $(V_n)_{n\in N}$, $V_1 = V$. The associated (F)-seminorm of $(V_n)_{n\in N}$ is given by: $q(x) = \inf\{\delta \mid x \in W_\sigma\}$ for all $x \in E$, where $W = \sum_1^n V_1 + \sum_1^\infty \varepsilon_k V_{k+1}$, $\delta = n + \sum_1^\infty \varepsilon_k 2^{-k} (n = 0 \text{ or } n \in \mathbf{N})$, $\varepsilon_k = 1$ for at most finitely many $k \in \mathbf{N}$ and $\varepsilon_k = 0$ otherwise [**1**, p. 11]. Since the subset W_δ is a solid t-neighbourhood of 0, it follows that q is a t-continuous Riesz (F)-seminorm. Conversely, if p is an arbitrary t-continuous Riesz (F)-seminorm, then $(V_n)_{n\in N}$ is a solid topological string, where

$$V_n = \{x \in E \mid p(x) < 1/2^n\}$$

COROLLARY 1. *If (E,C,T^0) is the l.t.R.s. from [**3**], where T^0 is the finest linear topology, then T^0 is determined by a family of all Riesz (F)-seminorms on (E,C).*

Similary as in [**9**] for the l.c.R.s. we say that the l.t.R.s. (E,C,t) is barrelled (resp. quasibarrelled, bornological,...)if (E,t) is barreled (resp. quasibarrelled, bornological, $\cdots$) in the category of l.t.s.'s.

We know from [**9** Proposition (11.2)(c)] that if (E,C,t) is an l.t.R.s. then the solid hull of each t-bounded subset of E is t-bounded. Therefore the question arises naturally whether the converse is true, namely: if t is a linear topology on (E,C) such that the solid hull of each t-bounded set in E is t-bounded, is (E,C,t) an l.t.R.s.? Example (3.15) from [**9**] shows that the answear is, in general, negative. If (E,t) is a bornological space, we have the following result:

PROPOSITION 2. *Let t be a linear topology on a Riesz space (E,C) such that (E,t) is bornological. If the solid hull of each t-bounded subset of E is t-bounded, then (E, C, t) is an l.t.R.s.*

Proof. We shale show that t is a linear solid topology. For this, let U be a t-neighbourhood of 0 in E. Since the solid hull of each t-bounded subset of E is t-bounded, then sk(U) absorbs all t-bounded subset of E. If $(U_n)_{n\in N}$, $U_1=U$, is a string which is generated by U, then $(\mathrm{sk}(U_n))_{n\in N}$ is a bornivorous string (this is easy to verify). Since (E,C,t) is a bornological space, we have that $(\mathrm{sk}(U_n))_{n\in N}$ is a topological string, i.e. sk(U) =sk$(U_1)\subset U$ is a solid t-neighbourhood of 0. Hence, (E,C,t) is an l.t.R.s.

An immediate consequence of Proposition 2 is the following:

COROLLARY 2. *The bornological space (E,C,t^β) associated with an l.t.R.s. (E,C,t) is always an l.t.R.s.* [**3**, *p.* 7].

If (E,t) is an arbitrary l.t.s., then there exists a linear topology t^b (resp. $t^{b^*},t^\beta,t^{lt},t^{bt}$)(see [**1**, pp. 32, 61, 70, 80 and **2**, p. 24]) which is generated by all closed (resp. closed bornivorous, bornivorous, locally topological, closed locally topological) strings in (E,t). It is known that an l.t.s. (E,t) is barrelled (resp. quasibarrelled, bornological, locally topological, ultra-b-barrelled) if and only if $t=t^b$ (resp. $t=t^{b^*},t=t^\beta,t=t^{lt},t=t^{bt}$).

PROPOSITION 3. *Let (E, C, t) be an l.t.R.s. Then (E, C,T^{b^*}) (resp. (E, C, t^{lt}), (E, C, t^{bt})) is an l.t.R.s.*

Proof. Let $(V_n)_{n\in N}$ be a closed bornivorous (resp. locally topological, closed locally topological) string in the space (E,C,t). Since the bounded sets in $(E,t),(E,t^{b^*})(E,t^{lt})$ and (E,t^{bt}) are the same, it follows that $(\mathrm{sk}(V_n))_{n\in N}$ is a closed solid bornivorous (resp. solid locally topological, closed solid locally topological) string in E, i.e. $(\mathrm{sk}(V_n))_{n\in N}$ is t^{b^*}-topological (resp. t^{lt}-topological, t^{bt}-topological). Hence, the spaces $(E,C,t^{b^*}),(E,C,t^{lt})$ and (E,C,t^{bt}) are l.t.R.s.'s if (E,C,t) is an l.t.R.s.

The following example shows that for the space (E,C,t^b) the conclusion of Proposition 3 is not true.

Example 1. Let $(E,C,\mathcal{P}_b)$ be an l.c.R.s., where $\mathcal{P}_b$ is the finest l.c. solid topology on a Riesz space (E,C) [**9**, p. 185], such that $(E,\mathcal{P}_b)$ is not barrelled.

Hence, $(E, C, \mathcal{P}_b)$ is not a barrelled l.c.R.s. It is known [**3**] that the l.c.s. $(E, \mathcal{P}_b)$ is the l.c.s. associated with (E, T^0), where T^0 is the finest linear solid topology on a Riesz space (E, C). Since $(E, \mathcal{P}_b)$ is not barrelled in the category of l.c.s.'s then (E, T^0) is not barreled in the category of l.t.s. [**1**, p.109]. Hence, $T^0 < (T^0)^b$. From this it follows that (E, C,$(T^0)^b$) it not an l.t.R.s.

It is know for each l.t.s. (E, t) there exists an l.t.s. (E, Rt) such that Rt is the coarsest linear topology which is finer that t and has the property R. In general, R is a property invariant under passage to an arbitrary inductive limit and the finest linear topology. For example, R is one of the properties being barrelled, quasibarrelled, ... [**1**, pp. 36, 61, 71, 73, 81, **4, 5, 6**]. Then, we say that Rt is the topology associated with an l.t.s. (E, t). If (E, C, t) is an l.t.R.s., the question asises naturally whether (E, C, Rt) is an l.t.R.s.? From Proposition 3 and Corollary 2, it follows that the answer is positive if (E, C, Rt) is the associated bornological (resp. locally topological) space. Example 1 shows that the answer to the question above is negative if (E, C, Rt) is the associated barrelled topology. For the quasibarrelled (resp. countably quasibarelled, ultra-b-barreled) associated space we have the following proposition:

PROPOSITION 4. *Let (E, C, t) be en l.t.R.s. Then space (E, C, t^{qt}) (resp. $(E, C, t^{cqt}), (E, C, t^{ubt})$ is an l.t.R.s., where t^{qt} (resp. $t^{cqt}; t^{ubt}$) is the quasibarrelled (resp. countably quasibarrelled, ultra-b-barrelled) topology associated with t.*

Proof. The proof follows by using Proposition 3, transfinite induction and [**3**, (1.2)] (see [**7**]).

If (E, C, t) is a bornological (resp. quasibarrelled) l.t.R.s., then any l-ideal in (E, C) is bornological (resp. quasibarrelled) with respect to the relative topology [**3**, p. 7]. Here we show that this is also true for ultra-DF (resp. countably quasibarrelled, locally topological, ultra$-b$-barrelled), ultra-D_b l.t.R.s.'s.

First, in terms of the order structure, we are able to give some characterizations of these l.t.R.s.'s:

PROPOSITION 5. *Let (E, C, t) be an l.t.R.s. and consider the following conditions:*

(i) (E, C, t) is countably quasibarrelled,

(ii) each solid bornivorous σ-barrel is topological,

(iii) (E, C, t) is locally topological,

(iv) each solid locally topological string is topological,

(v) (E, C, t) is ultra-b-barrelled,

(vi) each closed solid locally topological string is topological.

Then (i) $\Leftrightarrow$ (ii), (iii) $\Leftrightarrow$ (iv) and (v) $\Leftrightarrow$ (vi).

Proof. We known already that (i) $\Rightarrow$ (ii) holds. We shall show that (ii) implies (i). Let $(\cap_{n=1}^{\infty} V_n^{(j)})_{j \in N}$ be a bornivours σ-barrel where $(V_n^{(j)})_{n \in N}$ is a closed topological string for each $j \in \mathbf{N}$. Then $(\mathrm{sk}(V_n^{(j)}))_{n \in N}$ is a closed solid topological string and by fact that $\mathrm{sk}(\cap_{n=1}^{\infty} V_n^{(j)}) = \cap_{n=1}^{\infty} \mathrm{sk}(V_n^{(j)})$ it follows that $(\cap_{n=1}^{\infty} \mathrm{sk}(V_n^{(j)}))_{j\in N}$ is a solid bornivours σ-barrel, i.e. (ii) $\Rightarrow$ (i) holds, The proof that (iii)$\Leftrightarrow$ (iv) and (v) $\Leftrightarrow$ (vi) is similar.

The following result shoud be compared with corollaries (15.4) and (15.7) of [**9**].

Proposition 6. *Let $\mathcal{V} = (V_n)_{n\in N}$ be a solid topological (resp. solid bornivours, solid locally topological, closed solid locally topological) string in an arbitrary l-ideal F of an l.t.R.s. (E, C, t). Then there exists in (E, C, t) a string $\mathcal{U} = (U_n)_{n\in N}$ of the same type, such that $\mathcal{U} \cap F = \mathcal{V}$.*

Proof. Let $U_n = \{x \in E : y \in V_n$ whenever $0 \le y \le\mid x \mid$ and $y \in F\}$. It is clear that $U_n \cap F = V_n$ for each $n \in \mathbf{N}$. The proof that U_n is a t-neighbourhood of 0 (resp. closed, bornivorours) is the same as in [**8**] (resp. [**9**, pp. 181, 182]) for the locally convex case. It remains to show that $U_{n+1} + U_{n+1} \subset U_n (n = 1, 2, \dots)$ and that $\mathcal{U} = (U_n)_{n\in N}$ is a locally topological string in (E, C, t). For this, let $x = a + b$, where $a, b \in U_{n+1}$ and $y \in F$ such that $0 \le y \le\mid x \mid$. Now, we have that $0 \le y \le\mid a + b \mid\le\mid a \mid + \mid b \mid$. Since $[\ 0, \mid a \mid + \mid b \mid] = [0, \mid a \mid] + [0, \mid b \mid]$, it follows that $y = y_1 + y_2 \in V_{n+1} + V_{n+1} \subset V_n$. From this it follows that $x \in U_n$, i.e. $\mathcal{U} = (U_n)_{n\in N}$ is a string in E. We shall show that $\mathcal{U}$ is a locally topological string. Suppose, on the contrary, that U_n and a t-bounded solid subset B of E exist, such that $W \cap B \not\subset U_n \cap B$ for each $W \in \mathcal{W}$, where $\mathcal{W}$ denotes the family of all solid t-neighbourhoods of 0. Therefore there exists a $y_w \in W \cap B$ such that $y_w \notin U_{n_0} \cap B$, i. e. there is an $x_w \in F$ with $0 \le x_w \le\mid y_w \mid$ and $x_w \notin V_{n_0}$. Note that $\{y_w, W \in \mathcal{W}, \supset (f)\}$ is a t-bounded net in (E, C, t) converging to 0, i. e. by [**9**, Prop. (11.1)] the net $\{x_w : W \in \mathcal{W}, \supset\}$ converges to 0 in (E, C, t). But this is a contradiction with $x_w \notin V_{n_0}$, because $(V_n)_{n\in N}$ is a solid locally topological string in F. The proof of the proposition is complete.

By methods similar to those used in this proof, we can verify the following result:

Proposition 7. *Let F be an l-ideal of an l.t.R.s. (E, C, t) and $(V^{(j)})_{n\in N} = (\cap_{n=1}^{\infty} V_n^{(j)})_{j\in N}$ a solid bornivorous σ-barrel in F with respect to the relative topology. There exists a solid bornivorours σ-barrel $(U^{(j)})_{j\in N} = (\cap_{n=1}^{\infty} U_n^{(j)})_{j\in N}$ in (E, C, t) , such that $U^{(j)} \cap F = V^{(j)}$ is valid for each $j \in \mathbf{N}$.*

Corollary 3.*Any l-ideal in an ultra-DF (resp. countably quasibarrelled, locally topological, ultra-b-barrelled, ultra-D_b) l.t.R.s. is a space of the same type with respect to the relative topology.*

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Prirodno-matematički fakultet
34000 Kragujevac
Yugoslavia

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