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AN IMPROVED CONSTANT FOR THE MUNTZ-JACKSON THEOREM

H. N. Odogwu

Abstract. We improve a Newman result [**2**,**3**] from 1974 concerning approximation of a continuous function by generalized polynomials. He proved that every $f \in C[0,1]$ there exists a generalized polynomial $P(x) = \sum_{k=0}^{N} c_k x^{\lambda k}$ such that

$$| f(x) - P(x) | \leq A w_f(\varepsilon), \qquad x \in [0,1] \tag{1}$$

holds. Here $0 = \lambda_0 < \lambda_1 < \cdots \lambda_N$ are given numbers w_f is the modulus of continuity of f, $\varepsilon = \max\{| B(z)/z \mid \Re z = 1\}$, $B(z)$ is the Blaschke product corresponding to the above set of λ_k's and A is a constant. Newman [**2**] proved that (1) holds with $A = 368$. We show that (1) is valid with $A = 66$. We prove this by slightly modifying Newman's proof and choosing the size of an interval, to which a suitable contradiction is extended, optimally.

Muntz's theorem gives a necessary and sufficient condition for the linear hull of the power functions

$$\{x^{p_0}, x^{p_1}, x^{p_2}, \dots\} \ (1 = p_0 < p_1 < \dots, p_n \to \infty \text{ as } n \to \infty)$$

to be dense in $C[0,1]$. Jackon's theorem determines the rate of approximation of a continuous function by polynomials in terms of the modulus of continuity. Several authors tried to combine the two theorems. Let $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_N$ be given numbers, $\Lambda = \{\lambda_0, \lambda_1, \dots, \lambda_N\}$ and denote by P_* the set of all generalized polynomials, i. e.

$$P_* = \{P(x) = \sum_{k=0}^{N} C_k x^{\lambda_K} \mid \lambda_K \in \Lambda, \ C_k \text{ real}; \ k = 0, \dots N\}$$

How well can a function $f \in C[0,1]$ be approximated by elements of p_*? After several attempts the correct rate of approximation was found by Newman [**3**] in 1974.

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THEOREM 1(Newman [**3, 2**]). *For every $f \in C[0,1]$ there exists a generalized $P \in P_*$ such that*

$$| f(x) - P(x) | \leq A w_f(\varepsilon) \tag{1}$$

where A is a universal constant, w_f is the modulus of continuity of f and ε_* is given by

$$\varepsilon_* = \max_{\Re z - 1} | B_*(z)/z |$$

$B_*(z)$ being the Blaschke product corresponding to Λ, i.e.

$$B_*(z) = \prod_{k=1}^{N} \frac{z - \lambda_k}{z + \lambda_k} \tag{3}$$

In [**3**] Newman proved that (1) holds with $A = 100$. This proof however seems to contain some minor errors which influence the value of A.

Let $S = S[a,b]$ the class of real valued functions g defined on $[a,b]$ which satisfy $| g(x) - g(y) | \leq | x - y |$. Functions of S will be called contraction on $[a,b]$. It can be shown [**2**, p. 122] that $g \in S[a,b]$ if end only if g is absolutely continuous and $| g'(x) | \leq 1$ a. e. in $[a,b]$. In [**3**, p. 341] the formula (1) does not seem to hold with the constat 8 (but it certainly holds with 6π instead of 8). On p. 342 $N = 1/4\varepsilon$ is substituted. Since N is an integer there (degree of a polynomial) only $N = [1/4\varepsilon]$ can be taken, but for small values of N, $1/N = [1/4\varepsilon]^{-1}$ can increase to 8ε.

In [**2**] a slightly different proof, free of the above errors, was given for Thorem 1; however, the constant was increased to 368.

In this note we make an effort to reduce the value of A in (1).

THEOREM 2. *Theorem 1 is valid with $A = 66$.*

We follow the notations and proof of [**2**] with some changes. Let, for an $f \in C[0,1]$,

$$E_*(f) = \inf_{P \in P_*} \|f - P\|$$

be the distance of f from the subspace P_* and let

$$\rho_* = \sup_{g \in S[0,1]} E_*(g).$$

Instead of (1) we show that for every $g \in S = S[0,1]$:

$$E_*(g) \leq 33\varepsilon_*; \tag{4}$$

thus, $\rho_* \leq 33\varepsilon_*$. This implies (1) since by Theorem (1) of [**2**, p. 122] we have

$$E_*(f) \leq 2w_f(\rho_*) \leq 2w_f(33\varepsilon_*) \leq 66 w_f(\varepsilon_*). \tag{5}$$

To prove (4) let $g \in S[0,1]$, and extend g to the interval $[-a,a]$, $a > 1$ such that it remains a contraction on $[-a,a]$. We need the following

LEMMA 1. *There exists an algebraic polynomial P_M of degree $\leq M$ such that*

$$| g(x) - P_M(x) | \leq \frac{\pi^2}{2} \cdot \frac{a}{M+1}, \quad \textit{if } x \in [-a, a] \tag{6}$$

$$| P'_M(x) | \leq \frac{a}{\sqrt{a^2-1}} \quad \textit{if } x \in [-1, 1], \tag{7}$$

Proof. Let $f(u) = g(a \cos u)$, $u \in [-\pi, \pi]$. Then

$$| f(u+t) - f(u) | = | g(a \cos(u+t)) - g(a \cos u) | \leq a \mid \cos(u+t) - \cos u \mid \leq \mid t \mid .$$

Let further

$$\sigma_M(u) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(u+t) K_M(t) dt = \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) K_M(u-y) dy,$$

where $K_M(t) = 1/2 + \sum_{k=1}^{N} \rho_{M_k} \cos kt$, be such that $K_M(t) \geq 0$, $t \in [-\pi, \pi]$ and $\rho_{M1} = \cos \pi/M + 2$.

The existence of such a K_M is well-known [**1**, p. 337, Lemma 13.3.5]. Using Korovkin's estimate [**1**, pp. 335–336, Lemma 13.3.5] we get

$$\begin{aligned} | \sigma_M(u) - f(u) | &\leq \frac{1}{\pi} \int_{-\pi}^{\pi} | f(u+t) - f(u) | K_M(t) dt \\ &\leq \frac{1}{\pi} \int_{-\pi}^{\pi} a \mid t \mid K_M(t) dt \leq \frac{a\pi}{\sqrt{2}} \sqrt{1 - \rho_{M1}} \\ &= a\pi \mathrm{Sin} \frac{\pi}{2(M+2)} \leq \frac{\pi^2}{2} \frac{a}{M+1}. \end{aligned} \tag{8}$$

The function f is even; thus σ_M is a cosine polynomial and

$$P_M(x) = \sigma_M\left(\arccos \frac{x}{a}\right) = \frac{1}{\pi} \int_{-\pi}^{\pi} g\left[a \cos\left(\arccos \frac{x}{a} + t\right)\right] K_M(t) dt \tag{9}$$

is an algebraic plynomial of degree $\leq M$. From (8) we obtain (6) by substituting $u = \arccos x/a$. To prove (7) we notice that from (9)

$$P'_M(x) = -(a^2 - x^2)^{-1/2} \sigma'_M(\arccos x/a) \text{ a.e.}$$

Using (9) and the inequalities $| g'(x) | \leq 1$ a.e., $K_M(t) \geq 0$ we obtain

$$| P'_M(x) | \leq a(a^2 - x^2)^{-1/2} \leq a(a^2 - 1)^{-1/2} \quad \text{if} \quad x \in [-1, 1]$$

which proves (7).

Let us now return to the proof of Theorem 2. By (7) $\|P'_M\| C[-1, 1] \leq a(a^2 - 1)^{-1/2}$ therefore, applying Theorem 12 of [**2**, p. 21] for P'_M we get

$$| P_M^{(k)}(0) | \leq (M-1)^{k-1} \|P'_M\| < a(a^2 - 1)^{-1/2} M^{k-1}.$$

Since $P_M(x) = \sum_{k=1}^{M} \frac{P_M^{(k)}(0)}{k!} x^k$ by Lemmas 5, 6 of [**2**, p. 125] for every $k = 1, 2, \ldots$ there is a $q_k \in P_*$ such that

$$| x^k - q_k(x) | \leq | B(k) | \leq (\varepsilon_* k)^k \quad x \in [0,1].$$

With $Q(x) = \sum_{k=0}^{M} \frac{|P_M^{(k)}(0)|}{k!} q_k(x) \quad (q_0(x) = 1)$ we have $Q \in P_*$ and

$$\begin{aligned} | P_M(x) - Q(x) | &\leq \sum_{k=1}^{M} \frac{| P_M^{(k)}(0) |}{k!} (\varepsilon_* k)^k \\ &\leq \sum_{k=1}^{M} \frac{a}{\sqrt{a^2-1}} \frac{M^{k-1}}{k} (\varepsilon_* k)^k \ \text{ if } \ x \in [0,1]. \end{aligned} \tag{10}$$

The inequality (6) implies that

$$| g(x) - P_M(x) | < \frac{\pi^2}{2} \frac{a}{M+1} \ \text{ if } \ x \in [0,1], \tag{11}$$

thus, by the triangle inequality (10), (11), we obtain

$$E_*(g) \leq | g(x) - Q(x) | \leq \frac{\pi^2}{2} \frac{a}{M+1} + \frac{1}{M} \frac{a}{\sqrt{a^2-1}} \sum_{k=1}^{M} \frac{1}{k!} (M \varepsilon_* k)^k. \tag{12}$$

Lemma 2. (13) $k! \geq \sqrt{2k/3e}(k/e)^k$ *for* $k = 1, 2, \ldots$

Proof. Let A_k be the area bounded by the curves $y = \ln x$, $y = 0$, $x = 1.5$ and $x = k + 0.5$. Then

$$A_k = \int_{1.5}^{k+0.5} \ln x dx = \ln(k+1/2)^{k+1/2} - \ln(3/2)^{3/2} - (k-1)$$

and since the function $y = \ln x$ is concave, we have $A_k \leq T_k = t_2 + t_3 + \cdots + t_k t_n$ being the area bounded by $x = n - 1/2$, $x = n + 1/2$, $y = 0$ and $y - \ln x = 1/n(x - n)$ (the tangent line to $y = \ln x$ at $x = n$). We find that $T_k = \sum_{k=2}^{k} \ln n = \ln k!$ Using the inequality

$$(k+1/2)^k > \binom{k}{0} k^k + \binom{k}{1} k^{k-1} \cdot 1/2 = 3/2k^k$$

we easily obtain (13) from $A_k \leq T_k$.

By Lemma 2 our formula (12) can be written as

$$E_*(g) \leq \frac{\pi^2}{2} \frac{a}{M+1} + \frac{1}{e\sqrt{3/2}} \frac{a}{\sqrt{a^2-1}} \frac{1}{M} \sum_{k=1}^{M} (M \varepsilon_* k)^k. \tag{13}$$

Let

$$M = M(\varepsilon_*) = \begin{cases} [1/4\varepsilon_*], & \text{if} \quad 0 < \varepsilon_* \leq 1/8 \\ 1, & \text{if} \quad 1/8 < \varepsilon_* \end{cases}$$

First we show that

$$1/M + 1 < 4\varepsilon_*. \tag{14}$$

If $0 < \varepsilon_* \leq 1/8$, then $1/4\varepsilon_* - 1 \leq M \leq 1/4\varepsilon_*$; therefore $1/4\varepsilon_* < M + 1$ which implies the inequality (14). If $1/8 < \varepsilon_*$, Then $M = 1$ and again $1/4\varepsilon_* < 2 = M + 1$

Secondly we show that

$$\frac{1}{M}\sum_{k=1}^{M}(e\varepsilon_* M)^k \leq 5e/4 - e\varepsilon_* \tag{15}$$

We shall distinguish five cases.

Case 1. $0 < \varepsilon_* 1/20$ then $1/5\varepsilon_* \leq 1/4\varepsilon_* - 1 < M \leq 1/4\varepsilon_*$. Hence,

$$1/M < 5\varepsilon_* \text{ and } \sum_{k=1}^{M}(e\varepsilon_* k)^k \leq \sum_{k=1}^{M}\left(\frac{e}{4}\right)^k \leq \sum_{k=1}^{\infty}\left(\frac{e}{4}\right)^k - \frac{e}{4-e}$$

which implies (15).

Case 2. $1/20 < \varepsilon_* < 1/16$. Now $M = 4$ and $1/5\varepsilon_* = 20/5 = 4 < 1/4\varepsilon_*$. Hence $1/M < 5\varepsilon_*$ and, as in case 1, $\sum_{k=1}^{M}(e\varepsilon_* k)^k = e/4 - e$.

Case 3. $1/16 < \varepsilon_* < 1/12$. Now $M = 3$ and $\frac{1}{16/3\varepsilon_*} < \frac{16}{16/3} = M \leq 1/4\varepsilon_*$. Hence, $1/M < 16/3\varepsilon_*$ and $\sum_{k=1}^{M}(e\varepsilon_* k)^k \leq \sum_{k=1}^{3}(e/4)^k \leq 1.46$. Since $16/3.1.46 < 5e/4 - e$, (2) is again valid.

Case 4. If $1/12 < \varepsilon_* \leq 1/8$, then $M = 2$ and $1/6\varepsilon_* < 12/6 = M = 1/4\varepsilon_*$. Hence $1/M < 6\varepsilon_*$ and $\sum_{k=1}^{M}(e\varepsilon_* M)^k \leq \sum_{k=1}^{2}(e/4)^k \leq 1.15$. Since $6(1.15) < 5e/4 - e$, (2) is valid.

Case 5. $1/8 < \varepsilon$. Now $M = 1$, $1/M = 1$. On the other hand, $\sum_{k=1}^{M}(e\varepsilon_* M)^k = e\varepsilon_*$; thus, by $e < 5e/4 - e$, (2) is again valid. Thus for all values of $\varepsilon_* > 0$ we can use (1) and (2) to estimate $E_*(g)$. We obtain

$$E_*(g) \leq \left[\left(4 \cdot \frac{\pi^2}{2}\right)a + \sqrt{\frac{3}{2}}\frac{5}{4-e}\frac{a}{\sqrt{a^2-1}}\right]\varepsilon_*.$$

Now we would like to choose $a > 1$ such that the expression in the bracket has a minimal value. Denote the function in bracket by $h = h(a)$. Since

$$\lim_{a\to 1+0} h(a) = \lim_{a\to\infty} h(a) = +\infty, \quad h(a) > 0$$

and h is continuous, it a positive minimum for $a > 1$ The equation $h'(a) = 0$ gives

$$(a^2-1)^{3/2} = \sqrt{\frac{3}{2}}\frac{5}{4-e}\left[4 \cdot \frac{\pi^2}{2}\right]^{-2} \approx 0.2420442.$$

Hence, $a = a_1 \approx 1.1782978$. Since $h''(a) = \sqrt{3/2}5/4 - e3a(a^2-1)^{-5/2} > 0$, h assumes its minimum at $a = a_1$, $\min h(a) = h(a_1) = 32.391968 \leq 33$ and finally $E_*(g) \leq 33\varepsilon_*$. This is exactly (4) which implies (5), proving Theorem 2.

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Correspondence and Open Studies Institute
University of Lagos
Lagos, Nigeria.

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