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ON CONVERGENCE OF DERIVATIVES OF LINEAR COMBINATIONS OF MODIFIED LUPAS OPERATORS

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Abstract. We study some direct theorems in the simultaneous approximation by certain linear combinations of modified Lupas operators. We also consider a class of unbounded functions with growth of order of t^α.

1. Introduction

Motivated by Derriennic [**1**], Sahai and Prasad [**4**] proposed modified Lupas operators defined, for functions integrable on $[0,\infty)$ by

$$(L_n f)(x) = (n-1)\sum_{\nu=0}^{\infty} p_{n,\nu}(x)\int_0^\infty p_{n.\nu}(t)f(t)dt, \tag{1.1}$$

where

$$p_{n,\nu}(x) = \binom{n+\nu-1}{\nu} x^\nu (1+x)^{-(n+\nu)}.$$

It turns out that the order of approximation by these operators is at best $O(1/n)$, howsoever smooth the function may be. With the aim of bettering the said rate of approximation, May [**2**] and Rathore [**3**] have described a method for forming linear combination linear of positive operators. The approximation process follows.

$$L_n(f,k,x) = \sum_{j=0}^{k} C(j,k)L_{d_j n}(f;x), \tag{1.2}$$

where $d_0, d_1, d_2, \cdots d_k$ are arbitrary but fixed distinct positive integers. We define

$$C(j,k) = \prod_{\substack{i=0\\ i\neq j}}^{k} \frac{d_j}{d_j - d_i}, \quad k \neq 0 \text{ and } C(0,0) = 1, \tag{1.3}$$

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The object of the present paper is to study the problem of simultaneous approximation by the above linear combination of medified Lupas operators.

Throughout this paper $\langle a, b\rangle \subset [0,\infty)$ denotes an open interval containing the closed interval $[a, b]$. The superscript (r), $[\lambda]$ and $\|\cdot\|$ stand for the r-th derivative of the function, maximum integer not exceeding λ and the sup-norm on $[a, b]$ respectively.

2. Auxiliary results

We shall need the following results:

Lemma 2.1. [**4**]. *Let*

$$T_{n,m} = (n-r-1)\sum_{\nu=0}^{\infty} p_{n+r,\nu}(x)\int_0^{\infty} p_{n-r,\nu+r}(t)(t-x)^m dt.$$

Then

$$\begin{aligned} T_{n,0} = 1, \quad T_{n,1} =& \frac{(r+1)(1+2x)}{(n-r-2)}, \quad n > (r+2)\\ (n-m-r-2)T_{n,m+1} =& x(1+x)(T^{(1)}_{n.m} + 2mT_{n,m-1})\\ &+(m+r+1)(1+2x)T_{n,m}; \quad n > m+r+2. \end{aligned}$$

And hence $T_{n,m} = O(n^{-[(m+1)/2]})$.

Lemma 2.2. [**4**]. *For* $r = 0, 1, 2, \ldots$ *we have*

$$(L_n^{(r)} f)(x) = \frac{(n-r-1)!(n+r-1)!}{(n-1)!(n-2)!}\sum_{\nu=0}^{\infty} p_{n+r,\nu}(x)\int_0^{\infty} p_{n-r,\nu+r}(t)f^{(r)}(t)dt.$$

Lemma 2.3. [**2**]. *If* $C(j,k), j = 0, 1, 2\ldots k$ *are defined as in* (1.3), *then*

$$\sum_{j=0}^{k} C(j,k)d_j^{-m} = \begin{cases} 1 & m = 0\\ 0 & m = 1, 2, \ldots k. \end{cases}$$

3. Main results

Theorem 3.1. *Let* f *be integrable on* $[0,\infty)$ *admitting* $(2k+r+2)$*-th derivative at a point* $x \in [0,\infty)$ *with* $f^{(r)}(x) = O(x^{\alpha})$, *where* α *is a positive integer not less that* $2k+2$, *as* $x \to \infty$. *Then*

$$\lim_{n\to\infty} n^{k+1}[L_n^{(r)}(f,k,x) - f^{(r)}(x)] = \sum_{i=1}^{2k+2} Q(i,k,r,x)f^{(i+r)}(x), \tag{3.1}$$

$$\lim_{n\to\infty} n^{k+1}[L_n^{(r)}(f,k+1,x) - f^{(r)}(x)] = 0, \tag{3.2}$$

where $Q(i,k,r,x)$ are certain polynomials in x of degree at most i. Furthermore if $f^{(2k+r+2)}$ exists and is continuous on $\langle 1,b\rangle$ then (3.1) *and* (3.2) *hold uniformly on* $[a,b]$.

Proof. By Lemma 2.2. and Taylor's expansion of f, we are led to

$$\sum_{j=0}^{k} C(j,k)\frac{(d_jn-1)!(d_jn-2)!}{(d_jn+r-1)!(d_jn-r-2)!}L_{djn}^{(r)}(f;x)-f^{(r)}(x)$$

$$=\sum_{j=0}^{k} C(j,k)\left[(d_jn-r-1)\sum_{\nu=0}^{\infty}p_{djn+r,\nu}(x)\int_0^{\infty}p_{djn-r\nu+r}(t)\{f^{(r)}(t)-f^{(r)}(x)\}dt\right]$$

$$=\sum_{j=0}^{k} C(j,k)(d_jn-r-1)\sum_{\nu=0}^{\infty}p_{d_jn+r,\nu}(x)\int_0^{\infty}p_{d_jn-r\nu+r}(t)$$

$$\left\{\sum_{i=1}^{2k+2}\frac{f^{(i+r)}(x)}{i!}(t-x)^i+\varepsilon(t-x)(t-x)^{2k+2}\right\}dt$$

$$=\sum_{i=1}^{2k+2}\frac{f^{(i+r)}(x)}{i!}\sum_{j=0}^{k}C(j,k)(d_jn-r-1)\sum_{\nu=0}^{\infty}p_{d_jn+r,\nu}(x)$$

$$\int_0^{\infty}p_{d_jn-r,\nu+r}(t)(t-x)^i dt$$

$$+\sum_{j=0}^{k} C(j,k)(d_jn-r-1)\sum_{\nu=0}^{\infty}p_{d_jn+r,\nu}(x)\int_0^{\infty}p_{d_jn-r\nu+r}(t)\varepsilon(t-x)(t-x)^{2k+2}dt$$

$$=\sum_{i=1}^{2k+2}\frac{f^{(i+r)}(x)}{i!}\sum_{j=0}^{k}C(j,k)T_{d_jn,i}(x)+E_{n,r,k}(x),$$

where

$$\varepsilon(t-x)=(t-x)^{2k-2}\left(f^{(r)}(t)-\sum_{i=0}^{2k+2}\frac{f^{(i+r)}(x)}{i!}(t-x)^i\right)\quad\text{for}\quad t\neq x$$

$$=0,\quad\text{otherwise.}$$

Using Lemma 2.1 and 2.3,

$$\sum_{i=0}^{2k+2}\frac{f^{(i+r)}(x)}{i!}\sum_{j=0}^{k}C(j,k)T_{d_jn,i}(x)$$

$$=\sum_{i=0}^{2k+2}\frac{f^{(i+r)}(x)}{i!}\sum_{j=0}^{k}C(j,k)O\left(\frac{1}{(d_jn)^{[(i+1)/2]}}\right)$$

$$=n^{-(k+1)}\sum_{i=1}^{2k+2}Q(i,k,r,x)f^{(i+r)}(x),$$

where $Q(i,k,r,x)$ are certain polynomials in x of degree at most i.

To prove (3.1) it suffices to show that $n^{k+1}E_{n,r,k}(x)\to 0$ for sufficiently large n. For arbitrary $\varepsilon>0, A>0$, there exists a $\delta>0$ such that $\mid \varepsilon(T-X)\mid<\varepsilon$ for $x\leq A$ and $\mid t-x\mid<\delta$. Now

$$\begin{aligned}E_{n,r,k}(x)=&\sum_{j=0}^{k}C(j,k)(d_jn-r-1)\sum_{\nu=0}^{\infty}p_{d_jn+r,\nu}(x)\\&\left(\int_{|t-x|<\delta}p_{d_jn-r,\nu+r}(t)\varepsilon(t-x)(t-x)^{2k+2}dt\right.\\&+\left.\int_{|t-x|>\delta}p_{d_jn-r,\nu+r}(t)\varepsilon(t-x)(t-x)^{2k+2}dt\right)\\=&I_1+I_2\quad\text{(say)}.\end{aligned}$$

To estimate I_1, using Lemma 2.1 we get

$$\begin{aligned}\mid I_1\mid\leq&\sum_{j=0}^{k}\mid C(j,k)\mid(d_jn-r-1)\sum_{\nu=0}^{\infty}p_{d_jn+r,\nu}(x)\int_{|t-x|<\delta}p_{d_j-r,\nu+r}(t)\\&\qquad\mid\varepsilon(t-x)\mid(t-x)^{2k+2}dt\\<&\,\varepsilon\sum_{j=0}^{k}|C(j,k)|(d_jn-r-1)\sum_{\nu=0}^{\infty}p_{d_jn+r,\nu}(x)\int_0^{\infty}p_{d_jn-r,\nu+r}(t)(t-x)^{2k+2}dt\\=&\,\varepsilon\sum_{j=0}^{k}\mid C(j,k)\mid T_{d_jn,2k+2}(x)\\=&\,\varepsilon\sum_{j=0}^{k}\mid C(j,k)\mid O((d_jn)^{-k-1})\\=&\,\varepsilon O(n^{-k-1}).\end{aligned}$$

Finally,

$$\begin{aligned}I_2=&\sum_{j=0}^{k}C(j,k)(d_jn-r-1)\sum_{\nu=0}^{\infty}p_{d_jn+r,\nu}(x)\int_{|t-x|\geq\delta}p_{d_jn-r,\nu+r}(t)\varepsilon(t-x)(t-x)^{2k+2}dt\\=&\sum_{j=0}^{k}C(j,k)\left((d_jn-r-1)\sum_{\nu=0}^{\infty}p_{d_jn+r,\nu}(x)\int_{|t-x|\geq\delta}p_{d_jn-r,\nu+r}(t)t^{\alpha}dt\right)\\=&\sum_{j=0}^{k}C(j,k)\left((d_jn-r-1)\sum_{\nu=0}^{\infty}p_{d_jn+r,\nu}(x)\int_{|t-x|\geq\delta}p_{d_jn-r,\nu+r}(t)\cdot\right.\\&\qquad\left.\cdot\left(\sum_{i=0}^{\alpha}\binom{\alpha}{i}(t-x)^ix^{\alpha-i}\right)dt\right)\end{aligned}$$

$$=\sum_{j=0}^{k} C(j,k)\Bigg((d_jn-r-1)\sum_{\nu=0}^{\infty} p_{d_jn+r,\nu}(x)$$
$$\int_{|t-x|\geq\delta} p_{d_jn-r,\nu+r}(t)\frac{(t-x)^{2k+3}}{\delta^{2k+3}}\left(\sum_{i=0}^{\alpha}\tbinom{\alpha}{i}(t-x)^i x^{\alpha-i}\right)dt\Bigg)$$
$$=\sum_{j=0}^{k}\frac{C(j,k)}{\delta^{2k+3}}\sum_{i=}^{\alpha}\tbinom{\alpha}{i}x^{\alpha-i}\cdot O(T_{d_jn,2k+i+3}(x))=O\left(\frac{1}{n^{k+2}}\right)$$

and (3.1) follows

The assertion (3.2) can be proved along similar lines using $L_n((t-x)^i,\ k+1,\ x)=O(n^{-(k+2)}),\ i=1,2,\dots$ which follows from Lemma 2.3.

The last assertion follows due to the uniform continuity of $f^{(2k+r+2)}$ on $[a,b]$ (enabling δ to become independent of $x\in[a,b]$).

This completes the proof.

Remark. We may note here that $\frac{(d_jn-1)!(d_jn-2)!}{(d_jn+r-1)!(d_jn-r-2)!}\to 1$ as $n\to\infty$.

THEOREM 3.2. *Let $1\leq p\leq 2k+2$ and f be integrable on $[0,\infty)$. If $f^{(p+r)}$ exists and is continuous on $\langle a,b\rangle$ having the modulus of continuity $\omega_{f^{(p+r)}}(\delta)$ on $\langle a,b\rangle$ and $f^{(r)}(x)=O(x^\alpha)(\alpha$ is a positive integer $\geq p)$ then for n sufficiently large*

$$\|L_n^{(r)}(f,k,x)-f^{(r)}\|\leq \mathrm{Max}\{C_1n^{-p/2}\omega_{f^{(p+r)}}(n^{-1/2}),\ C_2n^{-(k+1)}\}$$

where $C_1=C_1(k,p,r)$ and $C_2=C_2(k,p,r,f)$.

Proof. For every $t\in[0,\infty)$ and $x\in[a,b]$ we have

$$f^{(r)}(t)=\sum_{i=0}^{p}\frac{f^{i+r}(x)}{i!}(t-x)^i+\frac{f^{(p+r)}(\xi)-f^{(p+r)}(x)}{p!}(t-x)^p \\ +h(t,x)\chi(t), \tag{3.3}$$

where ξ lies between t and x and $\chi(t)$ is the characteristic function of the set $[0,\infty)/\langle a,b\rangle$. The function $h(t,x)$ for $x\in[a,b]$ is bounded by $M\ t^\alpha\mid t-x\mid^p$, for some constant M. Using (3.3) we get

$$\sum_{j=0}^{k} C(j,k)\frac{(d_jn-1)!(d_jn-2)!}{(d_jn+r-1)1(d_jn-r-2)!}L_{d_jn}^{(r)}(f;x)$$
$$=\sum_{j=0}^{k} C(j,k)(d_jn-r-1)\sum_{\nu=0}^{\infty} p_{d_jn+r,\nu}(x)\int_0^\infty p_{d_jn-r,\nu+r}(t)f^{(r)}(t)dt$$

$$=\sum_{j=0}^{k} C(j,k)(d_jn-r-1)\sum_{\nu=0}^{\infty} p_{d_jn+r,\nu}(x)\int_0^\infty p_{d_jn-r,\nu+r}(t)\Bigg\{\sum_{i=0}^{p}\frac{f^{(i+r)}(x)}{i!}(t-x)^i$$
$$+\frac{f^{(p+r)}(\xi)-f^{(p+r)}(x)}{p!}(t-x)^p+h(t,x)\chi(t)\Bigg\}dt$$

$$=\sum_{j=0}^{k}C(j,k)(d_jn-r-1)\sum_{\nu=0}^{\infty}p_{d_jn+r,\nu}(x)\int_0^\infty p_{d_jn-r,\nu+r}(t)\cdot\sum_{i=0}^{p}\frac{f^{(i+r)}(x)}{i!}(t-x)^i dt+$$

$$+\sum_{j=0}^{k}C(j,k)(d_jn-r-1).$$

$$\cdot\sum_{\nu=0}^{\infty}p_{d_jn+r,\nu}(x)\int_0^\infty p_{d_jn-r,\nu+r}(t)\left(\frac{f^{(p+r)}(\xi)-f^{(p+r)}(x)}{p!}\right)(t-x)^p dt$$

$$+\sum_{j=0}^{k}C(j,k)(d_jn-r-1)\sum_{\nu=0}^{\infty}p_{d_jn+r,\nu}(x)\int_0^\infty p_{d_jn-r,\nu+r}(t)h(t,x)\chi(t)dt$$

$$=I_1+I_2+I_3 \text{ (say)}$$

An application of Lemma 2.1 gives us $I_1 = f^{(r)}(x) + O(n^{-(k+1)})$ uniformly in $x \in [a,b]$

To estimate I_2, for every $\delta > 0$, we have

$$\mid f^{(p+r)}(\xi)-f^{(p+r)}(x)\mid \leq \omega_{f^{(p+r)}}(\mid \xi - x\mid) \leq \omega_{f^{(p+r)}}(\mid t-x\mid)$$
$$\leq (1+\mid t-x\mid/\delta)\omega_{f^{(p+r)}}(\delta).$$

Hence

$$\mid I_2\mid \leq \frac{1}{p!}\sum_{j=0}^{k}\mid C(j,k)\mid (d_jn-r-1)\sum_{\nu=0}^{\infty}p_{d_jn+r,\nu}(x)\int_0^\infty p_{d_jn-r\nu+r}(t)$$
$$(1+\mid t-x\mid/\delta)\mid t-x\mid^p \omega_{f^{(p+r)}}(\delta)dt$$

$$=\frac{\omega_{f^{(p+r)}}(\delta)}{p!}\sum_{j=0}^{k}\mid C(j,k)\mid (d_jn-r-1)\sum_{\nu=0}^{\infty}p_{d_jn+r,\nu}(x)\cdot$$
$$\int_0^\infty p_{d_jn-r,\nu+r}(t)\cdot(\mid t-x\mid^p+\mid t+x\mid^{p+1}/\delta)dt.$$

Using Schwardz inequality for summation and then for integration we find that

$$\sum_{\nu=0}^{\infty}p_{d_jn+r,\nu}(x)\int_0^\infty p_{d_jn-r\nu+r}(t)\mid t-x\mid^p dt \leq \left\{\sum_{\nu=0}^{\infty}p_{d_jn+r,\nu}(x).\right.$$
$$\left.\left(\int_0^\infty p_{d_jn-r\nu+r}(t)\mid t-x\mid^p dt\right)^2\right\}^{1/2}$$
$$\leq\left\{\sum_{\nu=0}^{\infty}p_{d_jn+r,\nu}(x)1/(d_jn-r-1)\left(\int_0^\infty p_{d_jn-r\nu+r}(t)(t-x)^{2p}dt\right)\right\}^{1/2}. \quad (3.4)$$

It may be remarked that (3.4) is true when p is replaced by $p+1$ and consequently

$$| I_2 | \leq \frac{\omega_{f^{(p+r)}}(\delta)}{p!} \sum_{j=0}^{k} | C(j,k) | \left\{ (d_j n - r - 1) \sum_{\nu=0}^{\infty} p_{d_j n+r,\nu}(x) \cdot \right.$$
$$\left. \int_0^{\infty} p_{d_j n-r,\nu+r}(t)((t-x)^{2p} + (t-x)^{2(p+1)}/\delta)dt \right\}^{1/2}$$

$$= \frac{\omega_{f^{(p+r)}}(\delta)}{p!} \sum_{j=0}^{k} | C(j,k) | O\{T_{d_j n,2p} + \delta^{-1} T_{d_j n,2(p+1)}\}^{1/2}$$
$$= \frac{\omega_{f^{(p+r)}}(\delta)}{p!} \sum_{j=0}^{k} | C(j,k) | \{O((d_j n)^{-p}) + \delta^{-1} O((d_j n)^{-p-1})\}^{1/2}.$$

Choosing $\delta = n^{-1/2}$ we get $| I_2 | \leq C_1 n^{-p/2} \omega_{f^{(p+r)}}(n^{-1/2})$.

Finally, choosing a positive number η such that $| t-x | \geq \eta$ we get

$$| I_3 | \leq \sum_{j=0}^{k} | C(j,k) | O\left((d_j n - r - 1) \sum_{\nu=0}^{\infty} p_{d_j n+r,\nu}(x) \int_{|t-x| \geq \eta} p_{d_j n-r,\nu+r}(t) t^{\alpha} | t-x |^p \, dt \right)$$
$$= \sum_{j=0}^{k} | C(j,k) | O\left((d_j n - r - 1) \sum_{\nu=0}^{\infty} p_{d_j n+r,\nu}(x) \right.$$
$$\left. \int_{|t-x| \geq \eta} p_{d_j n-r,\nu+r}(t) \left(\sum_{i=0}^{\alpha} \binom{\alpha}{i} | t-x |^i \, x^{\alpha-i} \right) | t-x |^p \, dt \right)$$
$$= \sum_{j=0}^{k} | C(j,k) | O\left((d_j n - r - 1) \sum_{\nu=0}^{\infty} p_{d_j n+r,\nu}(x) \right.$$
$$\left. \int_{|t-x| \geq \eta} p_{d_j n-r,\nu+r}(t) \frac{| t-x |^{2m-p}}{\eta^{2m-p}} \left(\sum_{i=0}^{\alpha} \binom{\alpha}{i} | t-x |^i \, x^{\alpha-i} \right) | t-x |^p \, dt \right), \ m > k+1$$
$$= \sum_{j=0}^{k} \frac{| C(j,k) |}{\eta^{2m-p}} O\left((d_j n - r - 1) \sum_{\nu=0}^{\infty} p_{d_j n+r,\nu}(x) \int_0^{\infty} p_{d_j n-r,\nu+r}(t) \right.$$
$$\left. \left(\sum_{i=0}^{\alpha} \binom{\alpha}{i} | t-x |^{i+2m} \, x^{\alpha-i} \right) dt \right)$$
$$= \sum_{j=0}^{k} \frac{| C(j,k) |}{\eta^{2m-p}} \cdot O((d_j n)^{-m}) = C_3 n^{-m}$$

uniformly in $x \in [a,b]$. Combining the estimates of $I_1 - I_3$, we obtain the required result.

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