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ON CLOSE-TO-CONVEX FUNCTIONS

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Abstract. Well-known coefficient and length results for the class of univalent close-to-convex functions are extended to a subclass of close-to-convex functions of high order.

1. Introduction. In [**3**] Goodman introduced the class $K(\beta)$ of normalised analytic functions which are close-to-convex of order $\beta \geq 0$, i.e. $f \in K(\beta)$ if f is analytic in $D = \{z : | z | < 1\}$ and if there exists $\varphi \in K(0) = C$ the class of normalised convex functions, such that for $z \in D$,

$$\left| \arg \frac{f'(z)}{\varphi'(z)} \right| \leq \frac{\beta\pi}{2}.$$

When $0 \leq \beta \leq 1$, $K(\beta)$ consists of univalent functions, whilst if $\beta > 1$ f need not even be finitely valent.

Denote by $V_k, (k \geq 2)$ the class of locally univalent functions with bounded boundary rotation and by R_k the class of functions with bounded radial rotation. Then $\varphi \in V_k$ if, and only if, $z\varphi' \in R_k$ (see e.g. [**2**]). In [**5**] Noor considered the class T_k defined as follows:

Definition. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic and locally univalent in D. Then for $k \geq 2$, $f \in T_k$ if there is a function $\varphi \in V_k$ such that for $z \in D$,

$$Re \frac{f'(z)}{\varphi"(z)} > 0 \tag{1}$$

Clearly $T_2 = K(1)$, the class of close-to-convex functions and it is easily seen [**5**] that $T_k \subset K(k/2)$ for $k \geq 2$

For $f \in K(1)$, Clunie and Pommerenke [**1**] showed that for $n \geq 2$, $n \mid a_n \mid < (2 + \sqrt{2}) e\, M(n/(n+1))$, where $M(r) = \max_\theta \mid f(re^{i\theta}) \mid$ and the author [**7**] showed that $L(r) < AM(r) \log 1/(1-r)$, where $L(r)$ denotes the length of the image of $\{z : \mid z \mid = r\}$ by $f(z)$ and where A is an absolute constant. The object of the

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AMS Subject Classification (1980 *revision*): Primary 30 C 45 present paper is to extend these results to the class T_k. The question of whether the results remain valid in the winder class $K(\beta)$ for $\beta > 1$ remains open.

2. Results. THEOREM 1. *Let* $f \in T_k (k \geq 2)$, *with* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. *Then for* $n \geq 2$,

$$n \mid a_n \mid \leq 3\, ke\, M(n/(n+1)) \tag{2}$$

Proof. We modify the method of Clunie and Pommerenke [**1**]. From (1) write

$$zf'(z) = g(z)h(z), \tag{3}$$

so that $g \in R_k,\ h(0) = 1$ and $\Re\, h(z) > 0$ for $z \in D$.

Thus we can write $zf'(z) = 2g(z)\Re\, h(z) - g(z)\overline{h(z)}$. Now with $z = re^{i\theta}$,

$$\begin{aligned} na_n =& \frac{1}{2\pi r^n} \int_0^{2\pi} zf'(z)e^{-in\theta} d\theta \\ =& \frac{1}{\pi r^n} \int_0^{2\pi} g(z)\Re[h(z)]e^{-in\theta} d\theta - \frac{1}{2\pi r^n} \int_0^{2\pi} g(z)\overline{h(z)}e^{-in\theta} d\theta. \end{aligned}$$

Therefore

$$\begin{aligned} n \mid a_n \mid \leq& \frac{1}{\pi r^n} \int_0^{2\pi} \mid g(z) \mid \Re\,[h(z)] d\theta + \frac{1}{2\pi r^n} \left| \int_0^{2\pi} \overline{g(z)} h(z) e^{-in\theta} d\theta. \right| \\ =& I_1(r) + I_2(r) \quad \text{say} \end{aligned}$$

Since $\Re\ h(z) > 0$ for $z \in D$, (3) gives

$$\mid g(z) \mid \Re[h(z)] = \Re[zf'(z)e^{-i \arg\ g(z)}].$$

Thus integrating by parts

$$I_1(r) = \frac{1}{\pi r^n} \Re \int_0^{2\pi} f(z) e^{-1 \arg\ g(z)} d_\theta(\arg\ g(z)) \leq \frac{k}{r^n} M(r),$$

since

$$\int_0^{2\pi} \left| \Re \frac{zg'(z)}{g(z)} \right| d\theta \leq k\pi \tag{4}$$

For $I_2(r)$, we have from (3)

$$I_2(r) = \frac{1}{2\pi r^{2n}} \left| \int_0^{2\pi} z^{n+1} f'(z) e^{-2i \arg\ g(z)} d\theta \right|. \tag{5}$$

Let $f_n(z) = \int_0^z t^n f'(t) dt$. Then integrating by parts gives

$$\mid f_n(z) \mid \leq 2r^n\, M(r). \tag{6}$$

Finally integrating by parts in (5) shows that

$$I_2(r) = \frac{1}{\pi r^{2n}} \left| \int_0^{2\pi} f_n(z) e^{-2i \arg\ g(z)} \Re \frac{zg'(z)}{g(z)} d\theta \right| \leq \frac{2k}{r^n} M(r)$$

on using (4) and (6).

Choosing $r = n/(n+1)$ gives (2).

THEOREM 2. *Let $f \in T_k (k \geq 2)$. Then for $0 < r < 1$,*

$$L(r) \leq A(k)M(r)\log 1/(1-r),$$

where $A(k)$ is a constant depending only upon k.

Proof. With $z = re^{i\theta}$, (3) gives

$$L(r) = \int_0^{2\pi} \Big| zf'(z) \Big| d\theta \leq \int_0^r \int_0^{2\pi} \Big| g'(\rho e^{i\theta})h(\rho e^{i\theta}) \Big| d\theta d\rho + \int_0^r \int_0^{2\pi} \Big| g(\rho e^{i\theta})h'(\rho e^{i\theta}) \Big| d\theta d\rho = J_1(r) + J_2(r) \quad \text{say.}$$

Now $J_1(r) = \int_0^r \int_0^{2\pi} \mid f'(\rho e^{i\theta})H(\rho e^{i\theta}) \mid d\theta d\rho$, where $H(z) = \frac{zg'(z)}{g(z)}$. Thus

$$\begin{aligned} J_1(r) &\leq \int_0^r \left(\int_0^{2\pi} \mid f'(\rho e^{i\theta} \mid^2\right)^{\frac{1}{2}} \left(\int_0^{2\pi} \mid H(\rho e^{i\theta}) \mid^2 d\theta\right)^{\frac{1}{2}} d\rho \\ &\leq 2\pi \int_0^r \left(1 + \sum_{n=2}^{\infty} n^2 \mid a_n \mid^2 \rho^{2n-2}\right)^{\frac{1}{2}} \left(\frac{1+(k^2-1)\rho^2}{1-\rho^2}\right)^{\frac{1}{2}} d\rho \end{aligned} \tag{7}$$

where we have used the Cauchy-Schwartz inequality, Parseval's equality and Lemma 2 in [**5**].

If $f \in K(\beta),\ 0 \leq \beta \leq 1$, then f is univalent in D [**3**]. However for $\beta > 1,\ f$ need nor be finitely valent [**4**]. Thus to estimate the first expression in (7) we proceed as follows.

With $\rho = n/(n+1),$ (2) gives

$$\sum_{n=2}^{\infty} n^2 \mid a_n \mid^2 \rho^{2n-2} \leq 9k^2e^2M(\sqrt{\rho})^2 \sum_{n=2}^{\infty} \rho^{n-2}. \tag{8}$$

It follows immediately from the definition of T_k that the class T_k forms a subset of a linear-invariant family of order $k/2+1$. Using Lemma 2.6 of [**6**] we deduce that $M(\sqrt{\rho}) < 2^{k+2}M(\rho)/\sqrt{\rho}$. Thus from (7) and (8) we have $J_1(r) < A(k)M(r)\log 1/(1-r)$.

To estimate $J_2(r)$ we note that since $\Re\, h(z) > 0$ for $z \in D,\ \mid h'(\rho e^{i\theta}) \mid \leq 2\,\Re\, h(\rho e^{i\theta})/(1-\rho^2)$. Thus

$$J_2(r) \leq 2\int_0^r \int_0^{2\pi} \frac{\mid g(\rho e^{i\theta}) \mid \Re\, h(\rho e^{i\theta})}{1-\rho^2} d\theta d\rho \leq 2k\pi \int_0^r \frac{M(\rho)}{1-\rho^2} d\rho$$

as in the proof of Theorem 1. Combining the estimates for $J_1(r)$ and $J_2(r)$ gives Theorem 2.

Remark. The proof of Theorem 2 shows that in fact

$$L(r) \leq A(k) \int_0^r \frac{M(\rho)}{1-\rho} d\rho.$$

Thus if $f \in T_k$ and $M(r) < 1/(1-r)^{\alpha},\ \alpha > 0$, then $L(r) < A(k,\alpha)/(1-r)^{\alpha}$, where $A(k,\alpha)$ denotes a constant depending only upon k and α.

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