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NOTE ON GENERALIZING PREGROUPS

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Abstract. Let P be a pree which satisfies the first four axioms of Stallings' pregroup. Then the following three axioms are equivalent:

[K] If ab, bc and cd are defined, and $(ab)(cd)$ is defined, then $(ab)c$ or $(bc)d$ is defined.

[L] Suppose $V = [x, y]$ is reduced and suppose $y = ab = cd$ where xa and xc are defined. Then $a^{-1}c$ is defined.

[M] Suppose $W = [x, y, z]$ is reduced. Then W is not reducible to a word of length one.

1. Introduction. Let P be a *pree* that is, let P be a nonempty set with a partial operation $m : D \to P$ where $D \subset P \times P$. [We say pq is *defined* if $(p, q) \in D$ and we will usual denote $m(p, q)$ by pq]. The *universal group* $G(P)$ of the pree P is the group with the following presentation:

$G(P) = gp[P; z = xy$ where xy is defined and $z = m(x, y)]$.

In other words, the generators of $G(P)$ are the elements of P and the defining relations of $G(P)$ come from the partial operation m on P. A pree P is said to be *group-embeddable* if P can be embedded in its universal group $G(P)$. (See Rimlinger [**2**].)

Stallings in [**4**] defined a collection of press, called pregroups, which guarantees such an embedding. Specifically, a pree P is a *pregroup* if it satisfies the following five axioms:

$[P_1]$ There exists an identity element $1 \in P$ such that, for all $p \in P$, $1p$ and $p\,1$ are defined and $1p = p = p\,1$.

$[P_2]$ For each $p \in P$ there exists $p^{-1} \in P$ such that pp^{-1} and $p^{-1}p$ are defined and $pp^{-1} = p^{-1}p = 1$.

$[P_3]$ If pq is defined, then $q^{-1}p^{-1}$ is defined and $(pq)^{-1} = q^{-1}p^{-1}$.

$[P_4]$ Supposing ab and bc are defined, then $a(bc)$ is defined if and only if $(ab)c$ is defined, in which case the two are equal.

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[P_5] If ab, bc and cd are defined, then either $(ab)c$ or $(bc)d$ are defined.

Kusher and Lipschutz in [**1**] generalized Stallings' result by weakening his last axiom [P_5]. Specifically, they proved that a pree P is group-embedable if it satisfies axioms [P_1] through [P_4] and the following two axioms:

[Q_5] If a_1a_2, a_2a_3, a_3a_4 and a_4a_5 are defined, then at least one of $(a_1a_2)a_3$, $(a_2a_3)a_4$, $(a_3a_4)a_5$ is defined.

[K] If ab, bc and cd are defined and $(ab)(cd)$ is defined, then $(ab)c$ or $(bc)d$ is defined.

Example 1. Let A and B be groups which intersect in a subgroup H. The amalgam $P = A \cup_H B$ is the classical example of a pregroup. Here $G(P)$ is the free product of A and B with H amalgamated.

Example 2. Let A, B, C be groups where A and B intersect in a non-trivial subgroup H, and B and C intersect in a nontrivial subgroup K. Also, suppose $B = H \oplus K$. Consider the amalgam $P = A \cup_H B \cup_K C$. Then P is not a pregroup. For example, let $a \in A - H$, $h \in H$, $k \in K$, $c \in B - K$ where $h \neq 1$ and $k \neq 1$. Then ah, hk and kc are each defined, but neither $(ah)k$ nor $(hk)c$ is defined. On the other hand, P does satisfy axioms [P_1] through [P_4] and [Q_5] and [K]. Moreover, $G(P)$ is the tree product of the A, B and C with H and K amalgamated.

Example 3. Let $T_n = (A_i; H_{rs})$ be a tree graph of groups with vertices A_i with edges H_{st}, and with diameter n. (Here H_{st} is a subgroup of groups A_s and A_t). Let $P = \cup_i A_i$ be the amalgam of the groups in T_n. It is known (cf. Serre [**3**]) that P is group-embedable where $G(P)$ is tree product of the groups A_i with the semigroups H_{st} amalgamated.

The group-embeddable pree P in Example 3 is neither a pregroup nor satisfies axiom [Q_5]. However, it does satisfy axiom [K] and the following axiom:

[T_k] If a_1a_2, $a_2a_3, \ldots, a_{n+1}a_{n+2}$ are defined, then at least one of $(a_1a_2)a_3$, $(a_2a_3)a_4, \ldots, (a_na_{n+1})a_{n+2}$ iz defined.

The question of whether axioms [K] and [T_n], together with axioms [P_1] through [P_4], are sufficient to guarantee that a pree P is group-embeddable is still an open question. This paper concerns axiom [K] and the following two axioms:

[L] Suppose $V = [x, y]$ is reduced and suppose $y = ab = cd$ where xa and xc are defined. Then $a^{-1}c$ is defined.

[M] Suppose $W = [x, y, z]$ is reduced. Then W is not reducible to a word of length one.

Specifically, we prove the following theorem.

Main Theorem: *Let P be a pree which satisfies axioms [P_1] through [P_4]. Then axioms [K], [L] and [M] are equivalent.*

In other words, if P satisfies one of [K], [L], [M], then it satisfies all three axioms. (We emphasize that the pree P in the Main Theorem need not satisfy axiom [P_5], [Q_5] or [T_n] for [K], [L] and [M] to be equivalent).

2. Notation, Preliminaries. Throughout this section P denotes a pree which satisfies axioms $[P_1]$ through $[P_4]$.

Let $X = [x_1, x_2, \dots, x_n]$ be an n-tuple of elements of P. Then X is caled a *word of length n.* The word X is said to be *reduced* if no pair $x_i x_{i+1}$ is defined. On the other hand, if some $x_i x_{i+1}$ is defined, then $Y = [x_1, \dots, x_i x_{i+1}, \dots, x_n]$ is said to be obtained from X by an *elementary reduction.*

The *triple abc* is said to be defined if ab and bc are defined and either $(ab)c$ or $a(bc)$ is defined. (By $[P_4]$, if abc is defined, then $abc = (ab)c = a(bc)$.)

Suppose $X = [x_1, x_2, \dots, x_n]$ and $A = [a_1, a_2, \dots, a_{n-1}]$ are words such that each triple $a_{i-1}^{-1} x_i a_i$ iz defined (where $a_0 = a_n = 1$). Then the *interleaving* of X by A, denoted by $X * A$, is said to be defined and

$$X * A = [x_1 a_1,\ a_1^{-1} x_2 a_2, \dots, a_{n-1}^{-1} x_n]$$

We write $X * A * B$ for $(X * A) * B$.

A word X is said to be *reducible* to a word Z if Z can be obtained from X by a sequence consisting of interleavings and elementary reductions. [Observe that if X is reducible to Z then X and Z represent the same element in the universal group $G(P)$ of P.]

Stallings proved the following in [**4**]:

Lemma A. *Suppose P satisfies axioms $[P_1]$ through $[P_4]$. Then:*

(1) $(x^{-1})^{-1} = x$ *for every x in P.*

(2) *If ax is defined, then $a^{-1}(ax)$ is defined and $a^{-1}(ax) = x$. Dually, if xa is defined, then $(xa)a^{-1}$ is defined and $(xa)a^{-1} = x$.*

(3) *If xa and $a^{-1}y$ are defined, then xy is defined if and only if $(xa)(a^{-1}y)$ is defined; in which case $xy = (xa)(a^{-1}y)$.*

A word X in P is said to be *fully reduced* if X is reduced and $X * A_1 * A_2 * \cdots * A_m$ is reduced whenever defined. Every word X of length $n = 1$ is automatically reduced and fully reduced. Lemma $A(3)$ immediately implies:

Lemma B. $X = [x, y]$ *is reduced if and only if X is fully reduced.*

Example 4. Consider groups $A = F \oplus G$, $B = G \oplus H$ and $C = H \oplus F$, where F, G, H are nontrivial subgroups. Let $P = A \cup B \cup C$. Then P is a pree which satisfies axioms $[P_1]$ through $[P_4]$. Let $f_1, f_2 \in F, g \in G$, $h \in H$ be nontrivial elements. Then $X = [f_1 g^{-1}, gh, h^{-1} f_2]$ is a reduced word of length three. Let $A = [g, h^{-1}]$. Then $X * A = [f_1, 1, f_2]$ is reducible to a word $Z = f_1 f_2$ of length one. Thus, by our Main Theorem, P does not satisfy any of the axioms $[K]$, $[L]$ or $[M]$.

3. Proof of Main Theorem. First we show that $[K]$ is equivalent to $[M]$. Suppose $[M]$ does not hold. Then there exists a reduced word X of length three which is reducible to a word Z of length one. Thus there exist words $A_1, \dots, A_m$ and B such that:

(1) $Y = X * A_1 * \cdots * A_m$ is reduced (2) $Y * B$ is not reduced and, after an elementary reduction, $Y * B$ is reducible to Z.

Suppose $Y = [x, y, z]$ and $B = [a, b]$. Then $Y * B = [xa, a^{-1}yb, b^{-1}z]$ is not reduced. Say $(xa)(a^{-1}yb)$ is defined. By Lemma $A(3)$, $(xa)(a^{-1}yb) = x(yb)$. By Lemma B,

$$[(xa)(a^{-1}yb),\ b^{-1}z] = [x(yb),\ b^{-1}z]$$

is reducible to Z if and only if $(x(yb))(b^{-1}z)$ is defined. The 4-tuple

$$[x,\ yb,\ b^{-1},\ z]$$

satisfies the hypothesis of axiom $[K]$. If axiom $[K]$ holds then either

$$x(yb)b^{-1} = xy \quad \text{or} \quad (yb)b^{-1}z = yz$$

iz defined. This contradicts the fact that Y is reduced. Thus $[K]$ cannot hold. Accordingly, $[K]$ implies $[M]$.

On the other hand, suppose $[K]$ does not hold. Then there exist a, b, c, d such that ab, bc, cd and $(ab)(cd)$ are defined but neither $(ab)c$ nor $(bc)d$ are defined. By $[P_4]$, $a(bc)$ is not defined. Thus $X = [a, bc, d]$ is reduced. Let $A = [b, c^{-1}]$. Then

$$X * A = [ab,\ b^{-1}(bc)c^{-1}, cd] = [ab,\ a,\ cd]$$

is reducible to a word of length one. Thus $[M]$ does not hold. Accordingly, $[K]$ and $[M]$ are equivalent.

Next we show that $[K]$ and $[L]$ are equivalent. Suppose $[K]$ holds. Furthermore, suppose $V = [x, y]$ is reduced and suppose $y = ab = cd$ where xa and xc are defined. By Lemma $A(2)$, $c^{-1}(cd) = c^{-1}(ab)$ is defined. By $[P_3], c^{-1}x^{-1}$ is defined, and by Lemma $A(2), (c^{-1}x^{-1})x = c^{-1}$ is defined. Thus the 4-tuple

$$[c^{-1}x^{-1},\ x,\ a,\ b]$$

satisfies the hypothesis of axiom $[K]$. The second triple $xab = xy$ is not defined since V is reduced. Thus the first triple $(c^{-1}x^{-1})xa = c^{-1}a$ is defined. Thus $[K]$ implies $[L]$.

On the other hand, suppose $[L]$ holds. Suppose ab, bc, cd and $(ab)(cd)$ are defined, and $(ab)c$ is not defined. We need to show that $b(cd)$ is defined for $[K]$ to hold. Note that

$$X = [ab,\ c] = [ab,\ (cd)d^{-1}] = [ab,\ b^{-1}(bc)]$$

is reduced where $(ab)(cd)$ and $(ab)b^{-1}$ are defined. By axiom $[L], (b^{-1})^{-1}(cd) = b(cd)$ is defined. Thus $[L]$ implies $[K]$. Therefore $[K]$ and $[L]$ are equivalent.

Accordingly, $[K]$, $[L]$ and $[M]$ are equivalent and our main theorem is proved.

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