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ON p-GROUPS OF SMALL ORDER

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Abstract. We prove that if G is a finite non-abelain p-group of order p^n (p a prime number), ≤ 6, then the order of G devides the order of the group of automorphisms of G.

Introduction and notation

The conjecture "if G is a finite non-cyclic p-group of order p^n, $n > 2$, then the order of G divides the order of the group of automorphisms of G" has been an interesting subject of research for a long time. Although a great number of papers have appeared on this topic, the conjecture still remains open. However, it has been established for abelain p-groups [**14**], for p-groups of class two [**8**], for non-cyclic metacyclic p-groups, $p =\neq 2$ [**3**] and for some other classses of finite p-groups ([**4, 5, 6, 7, 11**]). In this paper we show that this conjecture is also true for all finite non-abelian p-groups of order $p^n, n \leq 6$ for every prime number p.

Throughout this paper, G stands for a finite non-abelain p-group, of order p^n(p a prime number), with commutator subgroup G' and center Z. The order of a group X is denoted by $\mid X \mid$. We taje the lower and the upper central series of G to be:

$$G=L_0 \subset L_1 = G' \supset L_2 \subset \cdots \supset L_c = 0 \text{ and } 1=Z_0 \subset Z_1=Z \subset Z_2 \subset \cdots \subset Z_c=G,$$

where c is the class of G. $P(G) = \{x^p \mid x \in G\}$ and $\mid X \mid_p$ is the greatest power of p which divides $\mid X \mid$.

The invariants of G/G' are taken to be:

$$m_1 \geq m_2 \geq \cdots \geq m_t \geq 1 \text{ and } \mid G/G' \mid = p^m.$$

The number t is the number of generators of G. We denote by $A(G), I(G)$, $A_c(G)$, the group of automorphisms, inner automorphisms, central automorphisms of G respectively. Hom (G, Z) is the group of homomorphisms of G into Z. The group G has maximal class c, if $\mid G \mid = p^n$ and $c = n - 1$. G is called a PN-group,

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if G has no non-trivial abelian direct factor. G is metacyclic if it has a normal subgroup H such that both H and G/H are cyclic.

First we give some results which we shall use very often throughout the proof of the theorem.

LEMMA 1. [**6**] *(i) If $G = H \times K$, where H is abelian and K is a PN-group, then*

$$| A_c(G) | = | A_c(K) | \cdot | A(H) | \cdot | \mathrm{Hom}(K,H) | \cdot | \mathrm{Hom}(H, Z(K)) | .$$

(ii) If G is a PN-group of class c and s is the number of invariants of Z, then

$$| A(G) |_p \geq p^{2s+c-1} \ \ and \ \ | A(G) |_p \geq | A_c(G) | \cdot p^{c-1}.$$

(ii) If G is a PN-group and $\exp(G/G') \leq | Z |$, then $| A_c(G) | \geq | G/G' |$.

(iv) If the Frattini subgroup $\Phi(G)$ of G is cyclic, then $| A(G) |_p \geq | G |$.

LEMMA 2. [**5**] *If $m_1 \geq m_2 \geq \cdots \geq m_t \geq 1$ are the invariants of G/G', then* $\exp G \leq p^{m_1+m_2(c-1)}$. *For $t = 2$, $\exp Z \leq p^{m_1+m_2(c-1)-2}$ and $Z_{c-1} \leq \Phi(G)$ where $\Phi(G)$ is the Frattini subgroup of G.*

LEMMA 3. [**2**] *If $m_1 \geq m_2 \geq \cdots \geq m_t' \geq 1$ are the invariants of G/G', then*

$$p^{m_2} \geq \exp L_1/L_2 \geq \exp L_2/L_3 \geq \cdots \geq \exp L_{c-1}/L_c.$$

For $t = 2$, L_1/L_2 is cyclic of order at most p^{m_2}.

Now we prove some usefull lemmas.

LEMMA 4. *Let $G = H \times K$, where H is abelian and K is a PN-group. Let A, B, C, D be as in Lemma 1 with $A = A_c(K)$, $B =$*Hom(K,H), *$C = A(H)$ and $D =$*Hom$(H, Z(K))$. *Then (i) $| A(G) | \geq | A(K)) | \cdot | B | \cdot | C | \cdot | D |$ and (ii) $| A(G) | \geq p | I(G) | \cdot | B | \cdot | C | \cdot | D |$.*

Proof. (i) Let $\tilde{A} = \{\bar{\theta} \mid \bar{\theta}(h,k) = (h, \theta(k)),\ h \in H,\ k \in K,\ \theta \in A(K)\}$. Then $\bar{\theta}$ is an automorphism of G for every $\theta \in A(K)$. So $\tilde{A} \leq A(G)$. Since $A_c(G) < A(G)$, we get that $\tilde{A} \cdot A_c(G) = \tilde{A}ABCD = \tilde{A}BCD \leq A(G)$. But $| \tilde{A} \cap A_c(G) | = A_c(K) | = | A |$ and so

$$\begin{aligned} | A(G) | \geq | \tilde{A} \cdot A_c(G) | = | \tilde{A} | \cdot | A_c(G) | / | A | = | \tilde{A} | \cdot | B | \cdot | C | \cdot | D | = \\ = | A(K) | \cdot | B | \cdot | C | \cdot | D | . \end{aligned}$$

(ii) $I(G) = G/Z \simeq K/Z(K) \simeq I(K)$ and by [**9**] $| A(K)/I(K) | \geq p$. Hence the result follows from (i).

LEMMA 5. *If G has order 2^n and class $c = n-2$, then $| A(G) |_2 \geq | G |$.*

Proof. Since G has class $c = n-2$, $| G/G' | \leq 2^3$. We may assume that $| Z | > 2$; otherwise, Lemma 4 (ii) gives $| A(G) |_2 \geq 2 \cdot | I(G) | = 2 | G/Z | =$

$2 \cdot 2^{n-1} = 2^n$. If $\mid G/G' \mid = 2^2$, then G has a maximal subgroup M which is cyclic [**2**]. So $\Phi(G)$ is cyclic, as $\Phi(G) < M$. Then by Lemma 1 (iv) the result follows. If $\mid G/G' \mid = 2^3, \exp G/G' \leq 2^2 \leq \mid Z \mid$, and by Lemma 1 (iii), $\mid A_c(G) \geq 2^3$. Then $\mid A(G)^2 \geq \mid A_c(G) \mid \cdot 2^{c-1} \geq 2^3 \cdot 2^{n-3} = 2^n$.

Now we prove our theorem.

THEOREM. *If G is a finite non-abelian group of order p^n, p a prime number and $n \leq 6$, then $\mid A(G) \mid_p \geq \mid G \mid$.*

Proof. By Lemma 4(i) we may assume that G is a PN-group. If $Z \mid = p$, Lemma 4 (ii) gives $\mid A(G) \mid_p \geq p \mid I(G) \mid = p \mid G/Z \mid = p^n$. By Theorem 1 in [**5**], if $n = 5$, then $\mid A(G) \mid_p \geq p^n$. So $n = 6$. If G has class 5, then $\mid G/G' \mid = p^2$, $\exp G/G = p$ and by Lemma 1 (ii) we get $\mid A(G) \mid_p \geq \mid A_c(G) \mid \cdot p^{c-1} \geq p^2 \cdot p^4 = p^6$. Therefore $c \leq 4$. For $c = 2, \mid A(G) \mid_p \geq \mid G \mid$ by [**8**], and so, $3 \leq c \leq 4$. If Z is non-cyclic and s is the number of invariants of Z, then $s > 1$, and Lemma 1 (ii) gives $\mid A(G) \mid_p \geq p^{2s+c-1} \geq p^6$, as $c \geq 3$. Finally, if $Z \leq \Phi(G)$, then there exists a maximal subgroup M of G such that $Z \leq M$. Then $G = MZ$. But $\mid A(M) \mid \geq p^5$, since $\mid M \mid = p^5$, and so, $\mid A(G) \mid_p \geq p \mid A(<) \mid \geq p^6$ from [**11**]. Therefore we may assume that:

— G is PN-group of order p^6,

— Z iz cyclic of order greater than p,

— $Z \leq \Phi(G)$ and

— $3 \leq c \leq 4$.

Consider the following cases:

(a) Take $c = 4$. Let $G = L_0 > L_1 > L_2 > L_3 > L_4 = 1$ be the lower central series of G. Since $\mid L_i/L_{i+1} \mid \geq p$ for all $i = 1,2,3$, we have $p^2 \leq \mid G/L_1 \mid \leq p^3$.

If $\mid G/L_1 \mid = p^2$, then it has type (p,p) and by Lemma 2, $\exp Z \leq p^2$. Also by Lemma 3, L_1/L_2 has order p and $\exp L_i/L_{i+1} = p$ for all $i = 1,2,3$. For $p = 2$, the result follows from Lemma 5. Therefore we may assume that $p \neq 2$. If $\mid Z \mid > p^2, Z$ is not cyclic, as $\exp Z \leq p^2$; a contradiction. Hence $\mid Z \mid = p^2$. Then $\mid G/Z_3 \mid = p^2, \mid Z_3/Z_2 \mid = p$, where $G = Z_4 > Z_3 > Z_3 > Z_1 = Z > Z_0 = 1$ is the upper central series of G. Since $L_1 \leq Z_3$ and $\mid L_3 \mid = \mid Z_3 \mid = p^4$ we get $L_1 = Z_3$. Also $\mid L_1/L_2 \mid = p$ and $L_2 \leq Z_2$ gives $L_2 = Z_2$. Hence $Z < L_2$. Let H be a normal subgroup of G of order p^3 and exponent p. Then $H < Z_3 = L_1$ and $\mid L_1/H \mid = p$. So $L_1/H \leq Z(G/H)$, which gives $L_2 = [G, L_1] \leq H$. Since $\mid L_2 \mid = p^3 = \mid H \mid$, we get $L_2 = H$, and so, $Z < L_2 = H$. Therefore, $\exp Z = p$ and Z is not cyclic; a contradiction. So G has no normal subgroup H of order p^3 and exponent p. Then G is matacyclic and the result follows by [**4**].

If $\mid G/L_1 \mid = p^3$, then $\exp G/L_1 \leq p^2 \leq \mid Z \mid$, and Lemma 1 (iii) gives $\mid A_c(G) \mid \geq p^3$. Then $\mid A(G) \mid_p \geq p^3 \cdot p^{c-1} = p^6$.

(b) Take $c = 3$. Let $G = L_0 > L_1 > L_2 > L_3 = 1$ be the lawer central series of G. Then $p^2 \leq \mid G/L_1' \mid \leq p^4$.

If $\mid G/L_1 \mid = p^2$, $\exp Z \leq p^{c-2} = p$ and Z is not cyclic; a contradiction. Hence $\mid G/L_1 \mid \geq p^3$ and so, $\mid A_c(G) \mid \geq p^3$ in all cases as G/L_1 is not cyclic.

Then $| A(G) |_p \geq | A_c(G) | \cdot | G/Z_2 | \geq p^3 | G/Z_2 |$. Therefore we may assume that $| G/Z_2 | = p^2$; otherwise the theorem holds.

Let $| G/L_1 | = p^3$. Then G/L_1 has either type (p^2, p) or (p, p, p). In the first case, Lemma 3 gives $| L_1/L_2 | = p$, $| L_2 | = p^2$ and $\exp L_2 = p$. Since $L_2 \leq Z, Z$ is not cyclic; a contradition. If G/L_1 has type (p, p, p) then $\exp(L_1/L_2) = \exp L_2 = p$, so that $\exp L_1 \leq p^2$. Also $L_1 = \Phi(G)$ and $Z \leq \Phi(G) = L_1$. Therefore, $\exp Z \leq p^2$, and we may assume that Z is cyclic of order p^2. Since $| G/L_1 | = p^3, | L_1/L_2 | \geq p$, we get that $| L_2 | \leq p^2$. If $| L_2 | = p$, then Z is not cyclic, as $L_2 \leq Z$ and $\exp L_2 = p$. Therefore we may assume that $| L_2 | = p$ and L_2 is the only subgroup of Z of order p. Since G/L_1 has type $(p, p, p), G$ can be generated by 3 elements α, b, c such that α^p, b^p, c^p are elements of L_1. But $| G/Z_2 | = p^2$. So we can chose α, b, c such that $G = \langle \alpha, b, c \rangle$, $c^p \in Z$, $c \in Z_2$. Then $[\alpha, c], [b, c]$ are elements of Z of order p, and so, $[\alpha, c], [b, c]$ are elements of L_2. Since $x^p \in Z_2$, for every $x \in Z_1$, we have that $[\alpha, b]^p \in L_2$. If $[\alpha, b]^p = 1, \exp L_1 = p$, and so Z is not cyclic, as $Z \leq L_1$. Let $[\alpha, b]^p \neq 1$. Then $L_2 = \langle [\alpha, b] \rangle$. But $L_1 = \langle [\alpha, b], [\alpha, c], [b, c], L_2 \rangle$ [**1**, Lemma 1.1] and so $L_1 = \langle [\alpha, b] \rangle$. Then L_1 is cyclic; a contradiction, as $| L_1 | = p^3$ and $\exp L_1 \leq p^2$.

Let $| G/L_1 | = p^4$. If $\exp(G/L_1) \leq p^2 \leq | Z |$, then $| A_c(G) \geq p^4$ (Lemma 1 (iii)), and so, $| A(G) |_p \geq p^4 \cdot p^{c-1} = P^6$. Therefore, we may assume that G/L_1 has type (p^3, p) and $| Z | = p^2$. Then $| L_1/L_2 | = p$ and G can be generated by two elements α, b such that $\alpha^{p^3} \in L_1$, $b^p \in L_1$ and $\alpha^{p^2} \notin L_1$, $b \notin L_1$. Also $L_2 \leq Z$ and L_2 is the only subgroup of G of order p. Since G/Z_2 is elementary abelain of order $p^2, \Phi(G) \leq Z_2$, and so, $\Phi(G) = Z_2$. But $L_1 Z \leq Z(Z_2)$ and $| Z_2/L_1 Z | = p$ gives that Z_2 is abelain. As $G = \langle \alpha, b \rangle$ and $\alpha^p \in Z_2$, $b^p \in Z_2$, we get $Z_2 = \langle [\alpha, b], b^p, \alpha^p, Z \rangle$. If $\alpha^p \in Z$, then $\alpha^{p^2} \in L_2 \leq L_1$; contradiction. Since Z_2 has order p^4 and $[\alpha, b] = b^p$ if and only if $b^p \in Z$, we have to assume that $b^p \in Z$. On the other hand, if $\alpha^{p^3} = 1$, then $\langle \alpha^{p^2} \rangle$ is the only subgroup of Z of order p, and so, $L_2 = \langle \alpha^{p^2} \rangle$. Then $\alpha^{p^2} \in L_1$ a contradiction. So $\alpha^{p^3} \neq 1$ and since Z is cyclic of order p^2 we get that $Z = \langle \alpha^{p^2} \rangle$, $L_2 = \langle \alpha^{p^3} \rangle$ and α has order p^4. Since G has order $p^6, b^p \notin \langle \alpha^{p^2} \rangle$. This contradiction proves the theorem.

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