PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 45 (59), 1989, pp. 27-31

LARGE SUBALGEBRAS OF A BOOLEAN ALGEBRA

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Abstract. A subalgebra $\mathbf{A}$ of a Boolean algebra $\mathbf{B}$ is *large* (in $\mathbf{B}$) if there exists a $b \in B$ such that the algebra generated by the set $A \cup \{b\}$ is the whole algebra $\mathbf{B}$. In this paper we give a complete description of large subalgebras of a Boolean algebra.

We shall assume for this paper a basic knowledge of the theory of Boolean algebras and Stone spaces and we shall freely use the Stone duality theorem (M. H. Stone [**4**]). For every Boolean algebra $\mathbf{B}$ we shall denote its Stone space by $\mathbf{SB}$. The space $\mathbf{SB}$ is a compact, Hausdorff space whose domain SB is the set of all ultrafilters of the Boolean algebra $\mathbf{B}$. The base of the topology of $\mathbf{SB}$ is the family of clopen sets of the form $S(b) = \{F \in SB \mid b \in F\}$, for all $b \in B$.

A proper subalgebra $\mathbf{A}$ of a Boolean algebra $\mathbf{B}$ is large (in $\mathbf{B}$) if there exist a $b \in B$ such that the algebra generated by $A \cup \{b\}$, which we shall denote by $\mathbf{A}(b)$, is equal to $\mathbf{B}$. All the elements of $A(b)$ are of the form $(u \wedge b) \vee (v \wedge b^C)$, for some $u, v \in A$.

The inclusion map $i : A \to B$ is a monomorphism of Boolean algebras and its Stone dual, denoted by $k : SB \to SA$, is an epimorphism; actually a continous surjection of toplogical spaces. Sets of the form $k^{-1}(x)$, where $x \in SA$, will be callded *fibries* over points of the Stone space $\mathbf{SA}$. We shall say that a fibre is nontrivial if it contains at least two points, and that points of $\mathbf{SA}$ are obtained by identification of points of their fibres. However, if $b \in B - A$, then $S(b)$ is not a union of fibres. It is a nonfibered set.

To get some acquaintance with the theory that will be used in this paper, we shall mention the following result (D. Makinson [**2**]):

Lemma 1. *If $\mathbf{A}$ is a subalgebra of a Boolean algebra $\mathbf{B}$ and $b \in B$ is such that $b \in B - A$, then there are two ultrafilters F and G of $\mathbf{B}$ such that $b \in F$, $b^C \in G$ and $F \cap A = G \cap A$.*

The writing of this paper was supported by a grant from International Research & Exchanges Board

AMS Subject Classification (1980): Primary 06 E 05

Proof. In terms of Stone spaces this means that there exists a point $x \in SA$ such that the fibre $k^{-1}(x)$ contains at least two elements and $S(b)$ *meets* $k^{-1}(x)$, i. e. they have a nonempty intersection and $k^{-1}(x)$ is not a subset of $S(b)$. This exactly means that $S(b)$ is a fibered set.

Since we are working with Stone spaces, the set $k[S(b)]$ is compact and hence closed. However, $S(b) \subset k^{-1}[k[S(b)]]$ and if it happent that two sets are equal then, since k is onto, we should have $k[S(b^C)] = k[S(b)]^C$. This means that $k[S(b)] \subset SA$ is clopen and hence for some $a \in A$, $k[S(b)] = S(a)$, which means that $a = b$, but this is a contraction. So, the inclusion $S(b) \subset k^{-1}[k[S(b)]]$ is proper and there exists an $x \in SA$ such that $S(b)$ itersects $k^{-1}(x)$ and $k^{-1}(x)$ is not a proper subset of $S(b)$. #

The following lemma shows that, in case of large subalgebras, every fibre has at most two points.

Lemma 2. *If* $\mathbf{A}$ *is a large subalgebra of a Boolean algebra* $\mathbf{B}$*, then every ultrafilter of* $\mathbf{A}$ *has at most two different extensions to an ultrafilter of* $\mathbf{B}$*.*

Proof. In terms of Stone spaces, every fibre of a point of SA has at most two points of SB.

Let $b \in B$ be such that $\mathbf{A}(b) = \mathbf{B}$. According to Lemma 1, it is enough to prove that there are no two different ultrafilters of $\mathbf{B}$ extending the same ultrafilter of $\mathbf{A}$ and containing $b \in B$.

If we assume that there are two different ultrafilters F and G of $\mathbf{B}$ such that $b \in F$, $b \in G$ and $A \cap F = A \cap G$ then there would be an $a \in B$ such that $a \in F$ and $a^C \in G$. In that case, since $\mathbf{A}(b) = \mathbf{B}$, there exist $u, v \in A$ such that $a = (u \wedge b) \vee (v \wedge b^C)$ and so $a^C = (u^C \wedge b) \vee (v^C \wedge b^C)$.

Since $a^C \in G$ and since, beeing an ultrafilter, G is a prime filter, at least one of the two disjuncts must belong to G.

If it is $(u^C \wedge b)$, then $u^C \in G$. But $b, a \in F$ and $(b \wedge a) = (u \wedge b)$ so that $u \in F$. Since $u \in A$ and $F \cap A = G \cap A$, this means that $u \in G$ which is impossible.

If $(v^C \wedge b^C) \in G$, then $b^C \in G$, and this is impossible too. #

However, the converse of Lemma 2 is not allways true. To see this we can take the cofinite algebra of natural numbers and its subalgebra whose Stone space is obtained by the identification of points $2n$ and $2n+1$, for all $n \in \omega$. Since there is no finite or cofinite subset of ω intersecting all sets of the form $\{2n, 2n+1\}$. The subalgebra mentioned above is not large and each of its ultrafilters has at most two extensions to an ultrafilter of the cofinite algebra (see Theorem 2).

It is not immediately clear whether every Boolean algebra does have a large subalgebra. This question was raised in [**1**]. That the answer is positive we shall show in the next theorem. Before the theorem, let us observe that given a Boolean algebra $\mathbf{B}$ and $b \in B$, different from zero or one, there allways exists a maximal subalgebra of $\mathbf{B}$ not containing $b \in B$. This it an immediate consequence of the fact Boolean algebras are closed for the unions of chains.

Theorem 1. *Let* $\mathbf{B}$ *be a Boolean algebra,* $b \in B$*, and* $\mathbf{A}$ *a maximal subalgebra of* $\mathbf{B}$ *such that* $b \in B - A$*; then* $\mathbf{A}(b) = \mathbf{B}$.

Proof. Assume that $\mathbf{A}(b) \neq \mathbf{B}$ and let $a \in B - A(b)$. According to Lemma 1, there are ultrafilters F and G of $\mathbf{B}$ such that $a \in F$, $a^C \in G$ and $A(b) \cap F = A(b) \cap G$.

We can suppose that $b \in F$ and $b \in G$, for otherwise, we can work with b^C, which doesn't change anything since $\mathbf{A}(b) = \mathbf{A}(b^C)$.

Let us observe that both $(b \wedge a)$ and $(b \wedge a^C)$ are not elements of A. Since, if it they were, then from $(b \wedge a) \in A$ we should have $(b \wedge a) \in G$ and so $a \in G$, which is impossible. From $(b \wedge a^C) \in A$ we should have $a^C \in F$ which is impossible too.

Since $\mathbf{A}$ is a maximal subalgebra of $\mathbf{B}$ not containing $b \in B$, we have that, $b \in A(b \wedge a)$ and $b \in A(b \wedge a^C)$. This means that there exist $u_1, v_1, u_2, v_2 \in A$ such that;

$$b = (u_1 \wedge (b \wedge a)) \vee (v_2 \wedge (b \wedge a)^C), \qquad b = (u_2 \wedge (b \wedge a^C)) \vee (v_2 \wedge (b \wedge a^C)^C)$$

or equivalenty

$$b = (u_1 \wedge b \wedge a) \vee (v_1 \wedge a^C) \vee (v_1 \wedge b^C), \qquad b = (u_2 \wedge b \wedge a^C) \vee (v_2 \wedge a) \vee (v_2 \wedge b^C).$$

Since b cannot contain anything from b^C (recall the Stone representation theorem) we have that $(v_1 \wedge b^C) = 0$ and $(v_2 \wedge b^C) = 0$ so that $v_1 \leq b$ and $v_2 \leq b$. From this we have

$$b = (u_1 \wedge b \wedge a) \vee (v_1 \wedge a^C), \qquad b = (u_2 \wedge b \wedge a^C) \vee (v_2 \wedge a).$$

The last equations give

$$(b \wedge a^C) = (v_1 \wedge a^C) \leq v_1 \leq b, \qquad (b \wedge a) = (v_2 \wedge a) \leq v_2 \leq b$$

which finally gives that $b = v_1 \vee v_2$. But this is not possible since $b \in A^C$ and $v_1, v_2 \in A$. #

The maximality condition, of the type of Theorem 1, is not necessary for the largeness of a subalgebra of a Boolean algebra. To see this is enough to take the free Boolean algebra $F(X)$ over the set X of at least two free generators. Omitting one generator $x \in X$ and taking the free algebra over $X - \{x\}$ we obtain a subalgebra which is large in $F(X)$ but is still far from beeing a maximal subalgebra not containing $x \in X$.

Theorem 2. *A subalgebra* $\mathbf{A}$ *of a Boolean algebra* $\mathbf{B}$ *is large iff there exists* $b \in B$ *such that every ultrafilter of* $\mathbf{A}$ *has at most two extension to an ultrafilter of* $\mathbf{B}$ *and if such two extensions exist, then exactly one of them contains* $b \in \mathbf{B}$.

Proof. In terms of Stone spaces, $\mathbf{A}$ is large in $\mathbf{B}$ iff there exists s $b \in B$ such that every fibre of $\mathbf{SA}$ is a two element set and $S(b)$ meets every non-trivial fibre in exactly one point.

If $\mathbf{A}$ is large in $\mathbf{B}$, then there exists a $b \in B$ such that $\mathbf{A}(b) = \mathbf{B}$. Again, we shall neglect the trivial cases. According to Lemma 2, every fibre of $\mathbf{SA}$ has shall neglect the trivial cases. According to Lemma 2, every fibre of **SA** has at most two points. If there is a nontrivial fibre which is not intersected by $S(b)$ in exactly one point, we can assume that both points of that fibre belong to $S(b)$. For otherwise, they belong to $S(b^C)$ which is the same thing. This means that there are two ultrafilters F and G of $\mathbf{B}$ such that $b \in F, b \in G$ and $A \cap F = A \cap G$. Since F and G are different ultrafilters, there exists a $c \in B$ such that $c \in F$ and $c^C \in G$. However, $c \in B$ does not belong to A. Since $\mathbf{A}(b) = \mathbf{B}$, there exist $u, v \in A$ such that

$$c = (u \wedge b) \vee (v \wedge b^C).$$

But then we should have $b \wedge c = u \wedge b$ so that $u \in F$. Since F and G agree on A, we have $u \in G$. This means that $(b \wedge c) \in G$ and so $c \in G$, which is a contradiction.

To prove the converse, suppose $\mathbf{A}(b) \neq \mathbf{B}$. In that case, there would exist an $a \in B$ which does not belong to $A(b)$. This implies that there is an ultrafilter F of $\mathbf{A}(b)$ which has two exstensions, say F_1 and F_2, to ultrafilter of $\mathbf{B}$ such that $a \in F_1$ and $a^C \in F_2$.

If $b \in F$, then we have two different extensions F_1 and F_2 of the ultrafilter $A \cap F$ of $\mathbf{A}$ both containing $b \in B$, which is not possible.

If $b^C \in F$, then its extension to an ultrafilter of $\mathbf{B}$ can be none of F_1 or F_2. So, the ultrafilter $A \cap F$ of $\mathbf{A}$ has three different extensions and this is a contradiction too. #

From Theorem 2, it is evident that large subalgebras arise from the identification of points of Stone spaces, Namely, for given Boolean algebra $\mathbf{B}$ we can construct a large subalgebra of $\mathbf{B}$ as follows:

Take a homeomorphism $h : C \to D$, where $C, D \subset SB$ are closed sets separated by a clopen subset of the Stone space $\mathbf{SB}$. Define $\mathbf{A}$ such that A is the set of all $a \in B$ for which the restriction of h to the set $C \cap S(a)$ is still a homeomorphism of sets $C \cap S(a)$ and $D \cap S(a)$. Actually, we have identified closed sets C, D of SB which gave us the Stone space of a large subalgebra of $\mathbf{B}$.

In the example following Lemma 2, we have seen that the existence cf at most two extensions of every ultrafilter of $\mathbf{A}$ is not sufficient for the largeness of $\mathbf{A}$ in $\mathbf{B}$. It seems that for sufficiency, some kind of completeness condition is required. The condition of relative completeness was already used by Žarko Mijajlović in the discussion of finite extensions of Boolean algebras (Žarko Mijajlović [**3**]). We recall that a subalgebra $\mathbf{A}$ of a Boolean algebra $\mathbf{B}$ is relatively complete if for every set $S \subset A$, the supremum of S exists in $\mathbf{A}$ iff the supremim of S exists in $\mathbf{B}$ and they are equal. It is easy to see that $\mathbf{A}$ is relativelly complete in $\mathbf{A}(b)$.

THEOREM 3. *Let* $\mathbf{A}$ *be a complete subalgebra of a complete Booloean algebra* $\mathbf{B}$. *Then* $\mathbf{A}$ *is a large subalgebra of* $\mathbf{B}$ *iff every ultrafilter of* $\mathbf{A}$ *has at most two extensions to an ultrafilter of* $\mathbf{B}$.

Proof. To prove the theorem it is enough to construct $b \in B$ such that $S(b)$ meets all nontrivial fibres over the points of the Stone space SA.

First we claim that there exists a closed $C \subset SB$ such that the restriction, say h, of the projection k to C is a continuous surjection and:

(1) C is minimal in the sence that there is no D, closed proper subset of C, such that the restriction of k to D is still surjective and

(2) $h : C \to SA$ is a homeomorphism.

To prove (1) we consider the set P of all closed $C \subset SB$ such that the restriction of k to C is a surjection, ordered by inclusion. If L is a chain in P, the intersection of every finite subset of L (beeing totally ordered by inclusion) meets every fibre of SA. Since fibres are compact sets, the intersection of the chain L meets every fibre . Applying Zorn's Lemma downwards to the ordered set P we obtain a minimal closed C such that the restriction of k to C is continuous and surjective.

To prove (2) it is enough to show that h is one-one and this is the point where the completeness of $\mathbf{A}$ comes in. We recall that beeing a complete Boolean algebra means that the corresponding Stone space is extremally disconnected, that is, the topological space in which the closure of every open set is clopen.

Suppose that $x, y \in C$ are distinct points such that $h(x) = h(y) = z$, for some $z \in SA$. Since C is a Hausdorff space, there exists a clopen U such that $x \in U$ and $y \in U^C$. The set C is compact and h is a continuous surjection, so that the sets $h[U]^C$ and $h[U^V]^C$ are disjoint open sets in the extremally disconnected space SA. This means that their closuers are clopen and disjoint. The complements of this closures are clopen and their union is SA.

In particular, $z \in SA$ belongs to at least one of the last two sets, say $z \in V$ and $V = cl(h[U^C]^C)^C$. The set $W = U \cap h^{-1}[V]$ is open and nonempty, since $x \in W$. From this we have that $C - W$ is closed and the restriction of h to $C - W$ is still a surjection on a closed proper subset of C, contradicting the minimallity of C.

Since h is a homeomorphism, C meets every nontrivial fibre of SA in exactly one point and so does its complement $B - C$. It is clear that there exists a minimal projection, say $g : D \to SA$, where D is a closed subset of SB, containg $B - C$ and intersecting all nontrivial fibres. The algebra $\mathbf{B}$ is complete, so that the closure of $B - C$ is a clopen subset of the Stone space SB and there exists a $b \in B$ such that $S(b) = cl(B - C)$. However, it is clear that $S(b)$ meets every nontrivial fibre since $B - C \subset S(b) \subset D$.#

The close the paper let us remark that we can easily generalise our result of Lemma 2, to show that for all $n \in \omega$ and all $b_1, \dots, b_n \in B$, $\mathbf{A}(b_1, \dots, b_n) = \mathbf{S}$ implies that every ultrafilter of $\mathbf{A}$ has at most 2^n different extensions to the ultrafilter of $\mathbf{B}$. This can be the basis of an analysis of subalgebras that have a finite extension to the whole algebra (see [**3**]).

We have shown that in the case of Boolean algebras one can say quite a lot about large subalgebras. However, large subalgebras can be defined for any kind of algebras. In the case of distributive lattices, can we use prime filters to say something about large sublattices? In the case of groups, even the maximality condition, like that of Theorem 1, does not seem to be of any help.

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(Received 18 10 1988)