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MAXIMAL CANONICAL GRAPHS WITH THREE NEGATIVE EIGENVALUES

Aleksandar Torgašev

Abstract. A connected graph G is called canonical if no two of its nonadjacent vertices have the same neighbours in G. Let $C(3)$ be the class of all nonisomorphic canonical graphs with exactly 3 negative eigenvalues (including also their multiplicities). In this paper we prove that the class $C(3)$ contains exactly 32 maximal graphs with respect to relation to be induced subgraph. The orders of these graphs run over the set $\{9, 10, 11, 12, 13, 14\}$.

Results

A connected graph G from the class $C(3)$ is called minimal if each of its induced subraphs does not belong te this class, i. e. it is disconnected or not canonical or it has less than 3 negative eigenvalues. In [**2**] it is proved that the class $C(3)$ contains exactly 14 non-isomorphic minimal graphs, which we denote by $H_1, \dots, H_{14}$. The set of all these minimal graphs is denoted by $\mathcal{H}$. The graphs $H_i (i = 1, \dots, 14)$ are displayed in the following table according to the upper-diagonal form of their adjacency matrices. The first number in the bracket is the ordinal number of the graph H_i, the second one is the number $\mid H_i \mid$ of vertices in $\mid H_i \mid$, while the third one is the number of edges of H_i. The orders of graphs H_i are respectively 4,5 and 6.

The first graph in this list is the complete graph K_4, and the seventh one is the path P_6 on 6 vertices.

Obviously, each graph $G \in C(3)$ contains at least one the graphs $H \in \mathcal{H}$ as its minimal (induced) subgraph. But the corresponding minimal graph is not unique in the general case. The relation to be an induced subgraph is denoted by $\subseteq$.

The following two facts concerning the class $C(3)$ and minimal graphs have been also proved in [**4**].

Let $G \in C(3)$ and H be a minimal graph of G. Then:

(i)—each vertex $x \in V(G) - V(H)$ is adjacent to at least one vertex $y \in V(H)$;

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(ii)— no two vertices $x, y \in V(G) - V(H)$ can be adjacent to the same set of vertices from $V(H)$.

Table1. Minimal graphs with 3 negative eigenvalues

(01.4.6)	1	11	111			
(02.5.5)	1	11	001	0001		
(03.5.6)	1	11	011	0001		
(04.5.6)	1	11	001	0011		
(05.5.7)	1	10	110	1011		
(06.6.5)	1	01	001	0001	00100	
(07.6.5)	1	01	001	0001	00001	
(08.6.6)	1	01	001	0001	10001	
(09.6.6)	1	01	101	0010	00010	
(10.6.6)	1	11	100	0100	00100	
(11.6.6)	1	01	100	0011	00001	
(12.6.7)	1	10	011	0010	00011	
(13.6.7)	1	01	101	0011	00100	
(14.6.8)	1	01	100	0101	00111	

Consequently, if S is an arbitrary nonempty set of vertices from $V(H)$ and $T_s = \{x \in V(G) - V(H) \mid x$ is adjacent to all vertices from S and nonadjacent to all other vertices from $V(H)\}$,

we have $\mid T_2 \mid \leq 1$. Thus, each of the sets $T_s(S \subseteq V(H))$ is empty or a singleton.

Using the method of forbidden subgraphs it is also shown in [**4**] which sets of the form $T_s(S \subseteq V(H))$ must be empty, for each particular graph $H \in \mathcal{H}$ and a graph $G \in C(3)$ such that $G \subseteq H$. Consequently, it is proved that $\mid G \mid \leq 18$ for each graph $G \in C(3)$.

Hence, to generate the class $C(3)$ or the set of all maximal graphs from this class, one can apply the method of extension of the graphs $H_i \in \mathcal{H}$ in each patricular case $i = 1, \dots, 14$.

Namely, in each of the cases $H = H_i (i = 1, \dots, 14)$ one can add to H_i all the possible sets of vertices of the form $T_{s_j}(S_j \subseteq V(H_i))$, and by computing the spectrum of the graph $H_i \cup \cup_j T_{s_j}$, one investigates all the possible cases related to induced subgraph $\cup_j T_{s_j}$ (connected or disconnected).

In view of the all previous results, this procedure is certainly finite. But although finite, it was of long duration and needed several hours of computer time.

Applying this procedure, we have generated all the graphs from the class $C(3)$ as well as all the maximal graphs from this class. The main result reads:

THEOREM *The class $C(3)$ contains exactly 1800 nonisomorphic graphs.*

The same class also contains exactly 32 *maximal graphs which are presented in Table* 2.

We note that the first number in each line is the ordinal number of a maximal graph, the second number is the number of its vertices, and the last is the number of its edges.

Table 2. Maximal canonical graphs with 3 negative eigenvalues

(01.09.14)	1	01	001	0001	00100	010011	1001010	10101000					
(02.09.14)	1	01	001	0001	00100	100100	1000110	10110001					
(03.09.15)	1	11	001	0011	10010	010010	0101000	10001001					
(04.09.15)	1	01	001	0001	00100	010011	1001010	10101100					
(05.09.16)	1	11	001	0011	10010	010100	0100110	10001010					
(06.09.17)	1	01	101	0010	00010	110011	1001100	01100101					
(07.09.18)	1	11	001	0011	10010	010011	0101010	10001011					
(08.10.19)	1	11	001	0001	10010	010100	0100110	00100011	100010101				
(09.10.19)	1	01	001	0001	00100	101011	0100111	10010101	000011000				
(10.10.21)	1	01	101	0010	00010	101011	0100110	01100110	100110001				
(11.11.22)	1	10	110	1011	01100	100000	0011001	00111110	010010000	0000110000			
(12.11.22)	1	01	001	0001	00100	011010	1010111	01001101	000011000	0001011000			
(13.11.23)	1	01	001	0001	00100	011010	1010111	01001101	100001101	0000110000			
(14.11.23)	1	01	001	0001	00100	011010	1000011	01001101	010100111	0000110001			
(15.11.24)	1	11	111	0011	11001	100110	0110011	10100000	000100001	0100000010			
(16.11.25)	1	10	110	1011	10000	010011	0000110	00010110	001101110	0110000110			
(17.11.26)	1	10	110	1011	01100	100000	0011001	00011010	010010110	1110000111			
(18.11.26)	1	10	110	1011	00001	001101	0100101	01100100	100001110	0011100011			
(19.11.28)	1	11	001	0001	10010	001001	0101111	00001110	010101101	1001101101			
(20.12.26)	1	11	111	0011	10100	010001	1100100	01100100	100000101	0010000000	00010100101		
(21.12.26)	1	10	110	1011	00001	001101	0110010	00010001	010010101	1110010010	10000000000		
(22.12.28)	1	11	111	0011	10100	011001	1100100	00010110	100110100	0010000011	01000100000		
(23.12.30)	1	11	111	0011	11001	000101	1000000	10100110	011001111	0100000110	00100110000		
(24.12.30)	1	11	111	0011	11001	000101	0100101	00100000	100110011	1010000000	01011000111		
(25.12.30)	1	11	111	1000	01000	011010	1010011	00111111	000110110	0010010000	10010010000		
(26.12.32)	1	11	111	0011	00100	010111	1001111	11001000	000100001	0100100101	10001010011		
(27.12.36)	1	10	110	1011	01101	001101	0111000	01001011	100001111	1110001010	00011101011		
(28.13.30)	1	11	111	0011	11001	100110	0110001	00010001	010100010	1000100000	01000000000	101001000101	
(29.13.37)	1	11	111	0011	11001	000101	0100101	10010101	011010101	0101100001	10001011011	001000000010	
(30.14.35)	1	11	111	1000	01000	011000	1010000	00100000	110000000	0001001110	10010011100	010100111000	0011100101010
(31.14.43)	1	11	111	0011	11001	000101	0100101	10010101	011010101	1000101101	00100000000	101010110100	0101010010101
(32.14.49)	1	11	111	0011	11001	000101	0100101	10010101	011010101	1000101101	10101011010	010101001011	0010011110101

Corollary 1. *A graph $G \in C(3)$ if and only if it is an induced subgraph of a graph from Table 2 and an induced overgraph of a graph $H \in \mathcal{H}$.*

Corollary 2. *For each graph $G \in C(3)$ we have $\mid G \mid \leq 14$.*

We indicate the possible minimal graphs of maximal graphs $M_1, \dots, M_{32}$ from Table 2. They are not unique in the general case. The notation $H_i \subseteq M_j$ for some $i \leq 14$ and some $j \leq 32$ will mean that H_i is isomorphic to the subgraph of M_j induced by its first $\mid H_i \mid$ vertices.

The following relations hold:

$H_1 \subseteq M_j (j = 15, 20, 22, 23, 24, 25, 26, 28, 29, 30, 31, 32)$,

$H_2 \subseteq M_j (j = 8, 19)$,

$H_4 \subseteq M_j (j = 3, 5, 7)$,

$H_5 \subseteq M_j (j = 11, 16, 17, 18, 21, 27)$,

$H_6 \subseteq M_j (j = 1, 2, 4, 9, 12, 13, 14)$,

$H_9 \subseteq M_j (j = 6, 10)$.

Finally, for each $m = 4, 5, \dots 14$, we denote by A_m the number of all non-isomorphic graphs from the class $C(3)$ whic have exactly m vertices. Then we have

$$A_4 = 1,\ \ A_5 = 6,\ \ A_6 = 43,\ \ A_7 = 170,\ \ A_8 - 372,\ \ A_9 = 499,$$
$$A_{10} = 404,\ \ A_{11} = 215,\ \ A_{12} = 72,\ \ A_{13} = 15,\ \ A_{14} = 3.$$

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Matematički faluktet
Studentski trag 16 a
11000 Beograd
Yugoslavia

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