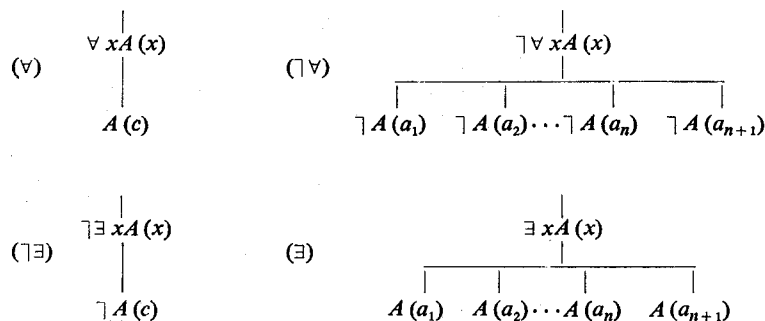


A PROOF PROCEDURE FOR THE FIRST ORDER LOGIC

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Abstract. A new proof procedure is given for the classical predicate logic which combines variants of the tableaux and resolution methods. Soundness and completeness of the resulting system are proved.

We shall present a proof procedure for the classical predicate logic combining features of the semantic tableaux method with the well known resolution rule of J. Robinson (see [2]). Given a closed first order formula (in a language without function symbols) whose validity is to be tested, we first build its *reduction tree* (growing downwards): put the formula at the top and then use the usual propositional tableaux rules (as described for instance in [1, pp. 26, 28]) as well as the following quantifier rules:



The rules apply as follows. A branch with $\forall xA(x)$ is extended by adding $A(c)$ as a new end node (the same for $(\neg\exists)$), where c is a new constant i. e. occurs in no formula on the tree. Similarly a branch containing $\exists xA(x)$ (the same for $(\neg\forall)$) gets a new extension $A(a_{n+1})$ where a_{n+1} differs from all a_1, \dots, a_n (by which this occurrence of $\exists xA(x)$ has been instantiated previously) and all $a_i (1 \leq i \leq n+1)$

occur already on the tree. If there are no constants on the tree at all then instantiate $\exists xA(x)$ by a new one. All (occurrences of all) formulas are used at most once w. r. t. a particular branch and discharged after that. As an exception an occurrence of $\exists xA(x)$ or $\neg\forall xA(x)$ is discharged only in case it cannot be instantiated by any constant anymore and no new constant can appear on the tree by any further reduction. By *basic sentences* of our language, enriched by new constants, we mean atomic sentences and their negations. The reduction trees we are most interested in are finished ones i. e. those on which all occurrences of all formulas are discharged. They have the fundamental property expressed by the following.

LEMMA. *Given a finished reduction tree T^* of a closed formula, the formula is valid iff for every model (in the language of T^*) there is a branch of T^* such that all basic sentences from that branch hold in the model.*

Proof. (\Leftarrow) Suppose that a given formula F is not valid i. e. there is an M such that $M \not\models F$. Define an expansion M^* of that model as follows. First enumerate all constants appearing in T^* into a sequence c_0, c_1, \dots and let $M_0 = M$. Then for all $n < \omega$ let the language $L(M_{n+1}) = L(M_n) \cup \{c_n\}$. Suppose that c_n was introduced by applying (\forall) to $\forall xA(x)$ (the same in case of $(\neg\exists)$). Then if $M_n \not\models \forall xA(x)$ take some $e \in |M|$ such that $M_n \not\models A[e]$ and put $c_n^{M_{n+1}} = e$. Otherwise $c_n^{M_{n+1}}$ is arbitrary. Finally let $|M^*| = |M|$, $L^* = L(M^*) = \bigcup_{n < \omega} L(M_n)$, $c_n^{M^*} = c_n^{M_{n+1}}$. Obviously $M^* \not\models F$ but suppose there is a branch \mathcal{B} of T^* whose all basic sentences hold in M^* . Then by induction it follows that all sentences from \mathcal{B} (including F) hold in M^* : if $A \wedge B \in \mathcal{B}$ then both $A, B \in \mathcal{B}$ (\mathcal{B} is finished!) and they hold in M^* by induction hypothesis (the same argument for $\neg\neg A, \neg(A \vee B), \neg(A \Rightarrow B)$). If $A \vee B \in \mathcal{B}$ then $A \in \mathcal{B}$ or $B \in \mathcal{B}$, say $A \in \mathcal{B}$, and again $M^* \models A$ by induction hypothesis (the same for $\neg(A \wedge B), A \Rightarrow B$). If $\forall xA(x) \in \mathcal{B}$ (the same for $\neg\exists xA(x)$) then $A(c_n) \in \mathcal{B}$ for some n and $M^* \models A(c_n)$ by induction hypothesis. Using the definition of $c_n^{M^*}$ we get $M^* \models \forall xA(x)$. Finally if $\exists xB(x) \in \mathcal{B}$ then again $B(c_n) \in \mathcal{B}$ for some n and $M^* \models B(c_n)$ by induction hypothesis. As $F \in \mathcal{B}$ we get the contradiction.

(\Rightarrow) Suppose that F is valid and let M be any structure in the language L^* . Take the restriction $M^C = M \upharpoonright \{e \in |M| \mid \text{for some constant } c, c^M = e\}$ and choose a branch \mathcal{B} by defining inductively its initial segments $\mathcal{B}_n = (F_0, F_1, \dots, F_n)$, $n = 0, 1, \dots$ (we identify a node of T^* with the formula placed at it): if F_n has only one successor take it for F_{n+1} . If it has at least two successors choose one (say the leftmost) that holds in M^C . We shall show by induction on n that $M^C \models F_n$ and the same time see that our definition is correct. Certainly $M^C \models F_0$, so suppose F_{n+1} is the only successor of F_n . It could only appear as a component of some F_i ($i \leq n$) of the form $\neg\neg A$ or $A \wedge B$ or $\neg(A \Rightarrow B)$ or $\neg(A \vee B)$ or $\forall xA(x)$ or $\neg\exists xA(x)$. Take for example the case $F_i = A \wedge B$, $F_{n+1} = A$. Then $M^C \models F_i$ by induction hypothesis so $M^C \models F_{n+1}$. If F_i is some $\forall xA(x)$ then for some c F_{n+1} is $A(c)$, so obviously $M^C \models F_{n+1}$ e. t. c. Suppose now that F_{n+1} has been chosen among two or more successors of F_n . Then it must be a component of some F_i of the form $A \vee B$ or $A \Rightarrow B$ or $\neg(A \wedge B)$ or $\exists xA(x)$, or $\neg\forall xA(x)$. If for instance F_i is

some $A \vee B$ and F_{n+1} is B then $M^C \models F_i$ by induction hypothesis, so $M^C \models A$ or $M^C \models B$. The choice of B implies that $M^C \not\models A$, hence M^C must satisfy B which shows both the possibility and correctness of our choice. The most interesting case is that of $\exists xA(x)$ (and $\neg\forall xA(x)$). If $M^C \models \exists xA(x)$ then $M^C \models A[e]$ for some $e \in |M^C|$. But $e = c^{M^C}$ for some c by the definition of M^C , so again there exists a successor of F_n true in M^C proving that F_{n+1} has been well defined. Finally, this shows that the whole of \mathcal{B} holds in M^C (including all basic sentences of \mathcal{B}) i. e. the lemma is proved, since M was chosen arbitrarily.

To get a complete set of rules we have to add the following *dual resolution rule* to the reduction rules given already:

$$\begin{aligned} & \text{Let } S_1 \text{ and } S_2 \text{ be sets of basic sentences, } A \in S_1 \text{ and } \neg A \in S_2; \text{ then} \\ \text{(DR)} \quad & \text{we resolve } S_1 \text{ and } S_2 \text{ to get their resolvent } R(S_1, S_2, A) = \\ & = (S_1 \setminus \{A\} \cup (S_2 \setminus \{\neg A\})). \end{aligned}$$

In derivations we allow only those S_i which are either sets of all basic sentences from a finished branch or resolvents obtained therefrom. A formula is proved if we can derive the empty set from it, using (DR) and the reduction rules.

As is well known, the existence of a model of a set of basic sentences is equivalent to the existence of the appropriate valuations where we take these sentences as propositional variables. If a valuation satisfies at least one of the sets of basic sentences from $R(S_1, S_2, A), S_3, \dots, S_k$ then it satisfies at least one from $S_1, S_2, S_3, \dots, S_k$. The proof systems from the tautology $(B \wedge C) \Rightarrow ((B \wedge A) \vee (C \wedge \neg A))$ since we can look at S_i 's as conjunctions of their members.

Since every valuation satisfies the empty set, it follows that every valuation satisfies at least one among the sets of basic sentences from the finished branches from which we derived the empty set. Together with (\Rightarrow) of Lemma this implies that our formula is valid, i. e. we can prove only valid formulas.

To show the converse take a valid formula and consider its finished reduction tree T^* . Then the set $\{\Gamma \mid \Gamma \text{ is the set of all basic sentences of some branch of } T^*\}$ has, by (\Leftarrow) of Lemma, the property that for any valuation at least one of its members is true. By the compactness theorem there is a finite subset $\{\Gamma_1, \dots, \Gamma_m\}$ with the same property. Using ideas from [3] and [2] we shall prove (by induction on $n = |\Gamma_1| + \dots + |\Gamma_m| - m$) that we can infer \emptyset from this subset ($|\Gamma_i|$ is cardinality of Γ_i).

If $n = 0$ then all Γ_i contain only one element so there are $j, k \leq m$ and an atomic sentence A such that $\Gamma_j = \{A\}$, $\Gamma_k = \{\neg A\}$. So the basis of induction is true.

For $n > 0$ take Γ_i with at least two elements and represent it as a disjoint union of nonempty sets Δ_1 and Δ_2 . By the induction hypothesis \emptyset can be inferred from $\Gamma_1, \dots, \Gamma_{i-1}, \Delta_1, \Gamma_{i+1}, \dots, \Gamma_m$. Repeating the same steps in this inference, with Δ_1 replaced by $\Delta_1 \cup \Delta_2 (= \Gamma_i)$ we get the inference of Δ_2 from $\Gamma_1, \dots, \Gamma_m$. Using induction hypothesis again we can extend this inference to an inference of \emptyset from $\Gamma_1, \dots, \Gamma_{i-1}, \Delta_2, \Gamma_{i+1}, \dots, \Gamma_m$ thus obtaining the inference of \emptyset from $\Gamma_1, \dots, \Gamma_m$.

This proves the *completeness* of our system: only valid formulas are provable.

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