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A REMARK ON THE PAPER "FIXED POINT MAPPINGS ON COMPACT METRIC SPACES"

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Abstract. We point out that the contractive condition for mappings considered in my paper [2] does not guarantee the existence of a fixed point and to indicate how it should be modified. So two fixed point theorems in a pseudocompact space are established, which are closely related to Theorems 1 and 2 from [2].

We shall prove a fixed point theorem in a pseudocompact Tychonoff space. A topological space X is said to be pseudocompact if every real valued continuous function on X is bounded. There are examples of pseudocompact spaces which are not compact. If X is a Tychonoff space, i.e. a completely regular Hausdorff space, then every real-valued continuous function on X is bounded and assumes its bounds.

THEOREM 1. *Let X be a pseudocompact Tychonoff space and let p be a symetric non-negative real valued continuous function over $X \times X$ such that $p(x,x)=0$ for all $x \in X$. If $T : X \to X$ is continuous and such that for all pairs of distinct $x, y \in X$ there exists a positive integer $n = n(x,y)$ such that*

$$(1)\quad p(T^n x, T^n y) < \max\{p(x,y), \min\{p(x,Tx), p(y,Ty), [p(x,Ty)+p(y,Tx)]/2\}\}$$

holds for all x, y for which the right hand side of the inequality (1) is positive, and $T^n x = T^n y$, if the right hand side of (1) is zero, then T has a unique fixed point.

Proof. Define on X a real-valued function F by $F(x) = p(x,Tx)$. Since F is continuous as composite of two continuous mappings, F assumes it bounds. Thus, there exists a point $u \in X$ such that

$$(2)\qquad F(u) = \min\{F(x) : x \in X\}.$$

We now show that T has a fixed point. If we suppose that for $x = u$ and $y = Tu$, the right hand side of the inequality (1) is positive, then we obtain

$$p(T^n u, T^{n+1} u) < \max\{p(u,Tu),\ \min\{p(u,Tu),$$
$$p(Tu,T^2 u),\ [p(u,T^2 u)+0]/2\}\} = p(u,Tu).$$

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So we have $F(T^n u) < F(u)$, which contradicts (2). Therefore, the right hand side of (1) for u and Tu is zero and so $T^n u = T^n Tu$. Hence $T^n u = TT^n u$, as $T^n Tu = TT^n u$. Thus we proved that $\nu = T^n u$ is a fixed point of T.

The uniqueness of a fixed point is easy to prove.

Since a compact metric space is a pseudocompact Tychonoff space, we have the following:

COROLLARY. *Let T be a continuous mapping of a compact metric space M into itself satisfying the inequality*

$$(3)\quad d(T^n x, T^n y) < \max\{d(x,y),\ \min\{d(x,Tx), d(y,Ty), [d(x,Tx)+d(y,Tx)]/2\}\}$$

for all x, y in M with $x \neq y$, where $n = n(x,y)$ is a positive integer. Then T has a unique fixed point.

Remark 1. This Corollary is one of possible correct variants of Theorem 1 from [2]. In [2] Theorem 1 is presented with the following contractive condition:

$$(3^*)\qquad d(T^n x, T^n y) < \max\{d(x,y), d(x,Tx), d(y,Ty), [d(x,Ty)+d(y,Tx)]/2\}.$$

The following counter-example shows that this contractive condition does not guarantee the existence of a fixed point.

Example. Let $M = \{1,2,4\}$ with the usual metric d and let T be a mapping of M onto itself such that $T(1) = 2$, $T(2) = 4$, $T(4) = 1$. Then T satisfies (3^*) with $n(1,2) = 3$, $n(1,4) = 1$ and $n(2,4) = 2$, but T is without fixed points.

By the same method of proof as presented in Theorem 1 it is easy to prove the following extension of Theorem 1:

THEOREM 2. *Let X be a pseudocompact Tychonoff space and let $p : X \times X \to R^+$ be a symmetric continuous function with $p(x,x) = 0$ for all $x \in X$. If $T\ X \to Y$ is continuous and such that for all distinct $x, y \in X$ there exists a positive integer $n = n(x,y)$ and a constant $C > 0$ such that*

$$(4)\qquad \begin{aligned} p(T^n x, T^n y) < \max\{p(x,y),\ [\min\{p(x,Tx), p(y,Ty)\} \\ + \min\{Cp(x,Tx), Cp(y,Tx)\}]\} \end{aligned}$$

for all x, y for which the right hand side of the inequality (4) *is positive and $T^n x = T^n y$, if the right hand side of* (4) *is zero, then T has a fixed point. If $C \leq 1$, then the fixed point is unique.*

Remark 2. Since the contractive condition in Theorem 2 in [2] is the same as in Theorem 1, it should be replaced with the contractive condition (3) of the Corollary above, or by the condition (4) with $p = d$. Theorem 3 in [2] should be deleted.

REFERENCES

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