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TEST OF SUPREMUM

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Abstract. To proceed from Bahadur's representation of the uniform sample quantile function, we created a new test for testing hypothesis $H_0(X : U(0;1))$.

1. Introduction. Let X_1, X_2,... be a sequence of independent random variables with the same distribution function $F(t)$. Suppose that:

1. $F(u)$ has two derivatives in a neighborhood of ξ_t, where $F(\xi_t) = t$ for $0 < t < 1$;
2. $F''(u)$is bounded there;
3. $f(\xi_t) = F'(\xi_t) > 0,\ 0 < t < 1$;
4. $F(0) = 0$ and $F(1) = 1$,

Let $Q_n(t) = X_{(k)}$, $(k-1)/n < t < k/n$ and $k = 1, 2, \dots, n$ be the quantile function, where $X_{(k)}$ is the sample k-th order statistic corresponding to sample $(X_1, X_2, \dots, X_n)$ and let $F_n(t)$ be the empirical distribution function. Bahadur [1] showed that

$$Q_n(t) = F^{-1}(t) - [F_n(F^{-1}(t)) - t]/f(F^{-1}(t)) + R_n(t),\ \ 0 < t < 1$$

and $R_n(t) = O(n^{-3/4}\log n)$. Kiefer [7] proved that

$$\limsup f(F^{-1}(t))R_n(t)/[32t(1-t)\log^3(\log n)/27n^3]^{1/4} = 1^1$$

with probability 1. Ghosh [5] obtained that $\sqrt{n}R_n(t)$ tends to zero when n tends to ∞. Duttweiler [4] showed that

$$E(R_n^2(l)) = f(F^{-1}(t))[t(1-t)]^{1/2}/n(n\pi)^{1/2}.$$

All in all, it is shown that $R_n(t)(\sqrt{u}R_n(t)$ and $f(F^{-1}(t))\sqrt{n}R_n(t))$ is a random noise whose influence becomes negligible as n tends to.

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[1]Throughout this paper sup is writen instead of $\sup_{0<t<1}$

Definition. The functional $G(F)$ is called *continuous-differentiable of order k in the point F_0* if there exists a functional $g(F_0,\omega)$ such that, for all sequences $\omega_\varepsilon \in D[0;1]$ (such that $\|\omega_\varepsilon - \omega\|$ tends to zero when ε tends to zero) the following hold: $[G(F_0+\varepsilon\omega_\varepsilon)-G(F_0)]/\varepsilon^k$ tends to $g(F_0,\omega)$ when ε tends to zero and $g(F_0,\omega_\varepsilon)$ tends to $g(F_0,\omega)$ when ε tends to zero.

THEOREM 1. *If the functional $G(F)$ is continuous-differentiable of order k in the point F_0, then $n^{k/2}[G(F_n)-G(F_0)]$ tends to $g(F_0,\omega^0)$ when n tends to ∞, where ω^0 is a Brownian bridge* [2].

2. Process $\mathbf{Z_n(t)}$. For the sample $(X_1X_2,\dots,X_n)$ with the same distribution function $F(t)$ which satisfy requirements 1, 2, 3 and 4, let us introduce the random process $Z_n(t)$:

$$\begin{aligned} Z_n(t) &= f(F^{-1}(t))[\sqrt{n}(F_n(t)-F(t)) - \nu n(Q_n(t)-F^{-1}(t))] \\ &= f(F^{-1}(t))[\beta_n(t)-\alpha_n(t)] \end{aligned}$$

where $\beta_n(t)=\sqrt{n}[F_n(t)]$ is an empirical process and $\alpha_n(t)=\sqrt{n}[Q_n(t)-F^{-1}(t)]$ is a sample quantile process.

THEOREM 2. *$Z_n(t)$ tends to process $B^*(t)$ (in distribution) where $B^*(t) = f(F^{-1}(t))B(F(t))+B(t)$ and $B(t)$ is a Brownian bridge.*

Proof. It is easy to see that:

$$\begin{aligned} Z_n(t) &= f(F^{-1}(t))\beta_n(t)+\beta_n(F^{-1}(t)) - f(F^{-1}(t))\sqrt{n}R_n(t) \ \text{ and} \\ |Z_n(t)-B^*(t)| &\le f(F^{-1}(t))|\beta_n(t)-B(F(t))| + |\beta_n(F^{-1}(t))-B(t)| \\ &\quad + f(F^{-1}(t))\sqrt{n}|R_n(t)| \end{aligned}$$

Theorem 4.3.1. in [3] implies that $\sup f(F^{-1}(t))|\beta_n(t)-B(F(t))|$ tends to zero (in distribution) when n tends to ∞. On the other hand: $\beta_n(F^{-2}(t)-B(t)) = \beta_n(F^{-1}(t)) - B(F(F^{-1}(t)))$ such that $\sup|\beta_n(F^{-1}(t))-B(t)|$ tends to zero (in distribution) when n tends to ∞. On the basis of Bahadur's, Kiefer's, Ghosh's and Duttweiler's considerations, we have: $\sup\sqrt{n}f(F^{-1}(t))|R_n(t)|$ tends to zero (in distribution) when n tends to ∞ and, consequently $\sup|Z_n(t)-B^*(t)|$ tends to zero (in distribution) when n tends to ∞.

COROLLARY. *When the sample is coming from uniform $U(0;1)$ distribution then $Z_n(t)=\sqrt{n}(F_n(t)-Q_n(t))$ tends to $2B(t)$ (in distribution) when n tends to ∞.*

THEOREM 3. *In case $U(0;1)$ r.v. $\mathrm{SUP}_n = \sup|Z_n(t)|$ tends (in distribution) to $2\,\mathrm{SUP}\,|B(t)|$ when n tends to a Brownian bridge.*

Proof. Let $G(F)=\mathrm{SUP}\,|(t)-F^{-1}(t)|$. We will show that the functional $G(F)$ is continuous-differentiable of order 1 in the point $F_0(t)=t$, $0<t<1$. First, $G(F_0)=0$ and second

$$G(F_0+\varepsilon\omega_\varepsilon) = \sup|t+\varepsilon\omega_\varepsilon-(t+\varepsilon\omega_\varepsilon)^{-1}| \cong \sup|t+\varepsilon\omega_\varepsilon-(t-\varepsilon\omega_\varepsilon)| = 2\sup|\varepsilon\omega_\varepsilon|$$

such that $[G(F_0+\varepsilon\omega_\varepsilon)-G(F_0)]/\varepsilon = 2\sup|\omega_\varepsilon|$ tends to $2\sup|\omega| = g(F_0,\omega)$ when ε tends to zero. Applying Theorem 1 we obtain:

$$\begin{aligned}\sqrt{n}[G(F_n)-G(F_0)] &= \sqrt{n}\sup F_n(t) - F_n^{-1}(t)| \cong \sqrt{n}\sup|F_n(t)-Q_n(t)| \\ &= \sqrt{n}\sup|Z_n(t)| \text{ tends to } 2\sup|\omega^0| = 2\sup|B(t)|\end{aligned}$$

(in distribution) when n tends to ∞.

3. Using the statistic $\mathrm{SUP}_{\mathbf{n}}$**.** For testing hypotesis $H_0(X : U(0;1))$ against $H_1(X : F(t) \neq t)$, we have the fact that:

$$P\{\sup|B(t)| > u\} = \sum_{k\neq 0}(-1)^{k+1}\exp(2k^2u^2), \quad u > 0$$

i.e. $\sup|B(t)|$ have distribution of Kolmoroff ([3], [4]).

Corollary. *For test-statistic* SUP_n*, we have critical region* $W_\alpha = (c_\alpha;\infty)$ *and critical value* c_α *is obtained for the condition* $P\{\mathrm{SUP}_n > c_\alpha/H_0\} = \alpha$.

When n tends to ∞, $c_\alpha = 2d_{n,\alpha}$, where $d_{n,\alpha}$ is a critical value for test Kolmogoroff. 'Noise' $\sqrt{n}R_n(t)$ for smaller value of n has an efect on critical value and we can obtain c_α by the help of the method Monte-Carlo. For example:

	c_α	
n	$\alpha = 0.05$	$\alpha = 0.01$
100	2.35	2.94
200	2.65	3.15
500	2.70	3.20
1000	2.7162	3.216

(BBC-micro computer)

Also, by the method Monte-Carlo we can investigate power of this *test of supremum* for different alternatives and compare the power of our test and the power of same other tests.

4. *Example.* For the class of alternative:

$$\{F(t;\theta) = [[(t+\theta)^2-\theta^2]/(2\theta+1)]^{1/2}, \ 0<t<1 \text{ and } 0<\theta<\infty\}$$

we compared the power of test of supremum and the test of Moran and obtained for $\alpha = 0.05$, $n = 100$ and number of samples $k = 2000$ (BBC-micro):

5. One-sided test. We can consider one-sided test of supremum. Let $\mathrm{SUP}^+ = \sup Z_n(t)$. In case $U(0;1)$ r.v. it is easy to see that SUP^+ tends to $2\sup B(t)$ (in distribution) when n tends to ∞. Because it is $P\{\sup B(t) > u\} = \exp(-2u^2)$, $u > 0$ [7], for testing hypothesis $H_0(X : U(0;1))$ against $H_1(X : F(t) \neq t)$, critical region is $W_\alpha = (c_\alpha;\infty)$ and, when n tends to ∞, $c_\alpha = (-2\ln\alpha)^{1/2}$.

θ	power of test supremum	power of test Moran
0.002	0.05	0.05
0.005	0.05	0.05
0.01	0.053	0.052
0.02	0.063	0.056
0.03	0.079	0.057
0.04	0.087	0.058
0.05	0.096	0.067
0.06	0.112	0.072
0.07	0.137	0.078
0.08	0.145	0.104
0.09	0.182	0.113
0.10	0.216	0.125
0.15	0.323	0.130
0.20	0.393	0.149
0.25	0.428	0.172
0.30	0.641	0.205

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