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A REMARK ON SUBSETS OF FREE BOOLEAN ALGEBRAS

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Abstract. We prove that every subalgebra of cardinality k, cf $k > \omega$, of a given free Boolean algebra contains an independent set of the same cardinality. As a corollary of this theorem we get, using the Stone duality, some well known results about Cantor spaces.

A Boolean algebra (BA) is a structure of the form $(A, \wedge, \vee, ', 0, 1)$. T will denote the theory of Boolean algebras. Terms in the language of Boolean algebras are called Boolean terms. A set X of elements of a given BA A is independent if for every finite subset Y of X and every mapping $f : Y \to \{0, 1\}$, $\wedge\{a^{f(a)} \mid a \in Y\} \neq 0$, where $a^0 = a'$ and $a^1 = a$. BA generated by an independent subset is a free BA. Obviously, it has the same cardinality as the set of generators. For a Boolean term $\varphi(x_0, \dots , x_{n-1})$ we will say that it is a canonical form if $\varphi(x_0, \dots , x_{n-1}) = \vee\{\wedge\{x_i^{f(i)} \mid 0 \leq i < n\} \mid f \in S_\varphi\}$ where $S_\varphi \subset^n \{0, 1\}$, and there is no proper subset $\{y_0, \dots , y_{n-1}\}$ of $\{x_0, \dots , x_{n+1}\}$ and Boolean term $\psi(y_0, \dots , y_{n-1})$ such that $T \models y \leftrightarrow \psi$. If $\varphi(x_0, \dots , x_{n-1})$ is a canonical form then its complementary form is $\varphi'(x_0, \dots , x_{n-1}) = \vee\{\wedge\{x_i^{f(i)}(0 \leq i < n\} \mid f \in^n \{0, 1)\backslash S_\varphi\}$ and $S_{\varphi'} =^n \{0, 1\}\backslash S_\varphi$. Obviously, $T \models \varphi \wedge \varphi' = 0$ & $\varphi \vee \varphi' = 1$.

LEMMA 1. *Let $\varphi(x_0, \dots , x_{n-1})$ be a canonical form, $\varphi'(x_0, \dots , x_{n-1})$ its complementary from and $m < n$. Then there is an $f \in S_\varphi$ and a $g \in S_\varphi$ such that $f \uparrow m = g \uparrow m$.*

Proof. Suppose, to the contrary, that for every $f \in S_\varphi$, $g \in S'_\varphi$ we have $f \uparrow m \neq g \uparrow m$. Then for every $f \in S_\varphi$ all $g \in^n \{0, 1\}$ such that $f \uparrow m = g \uparrow m$ belong to S_φ. Let $S^*_\varphi = \{f \uparrow m \mid f \in S_\varphi\}$. Then

$$T \models \varphi(x_0, \dots , x_{n-1}) = \vee\{\wedge\{x_i^{f(i)} \mid 0 \leq i < n\} \mid f \uparrow m = g,\ g \in S^*_y\}$$
$$T \models \varphi(x_0, \dots , x_{n-1}) = \vee\{\wedge\{x_i^{f(i)} \mid 0 \leq i < n\} \mid g \in S^*_y\}$$

contrary to our assumption that $\varphi(x_0, \dots , x_{n-1})$ is a canonical form.

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A family $\mathcal{B}$ is a Δ-system if $\exists\rho\forall x\in\mathcal{B}\forall y\in\mathcal{B}\ (x\neq y\Rightarrow x\wedge y=\rho)$; ρ is the root of the Δ-system $\mathcal{B}$.

Proposition 2. *Let θ be a regular cardinal, χ infinite and $\theta>\chi$, $\alpha^{<\chi}<\theta$ for $\alpha<\theta$. Let $|\mathcal{A}|\geq\theta$ and $\forall x\in\mathcal{A}(|x|<\chi)$. Then there exists a $\mathcal{B}\subset\mathcal{A}$ such that $|\mathcal{B}|=\theta$ and $\mathcal{B}$ forms a Δ-system.*

Proof. [Ku].

Proposition 3. *Let $|\mathcal{A}|\geq\theta$, $cf\theta\geq\omega$ and $\forall x\in\mathcal{A}(|x|<\omega)$. There exists a $\mathcal{B}\subset\mathcal{A}$ such that $\mathcal{B}$ is a Δ-system and $|\mathcal{B}|=\theta$.*

Proof. Using the same ideas as in the proof of Proposition 2, or presenting θ sa a sum of $\mathrm{cf}\,\theta$ many regular cardinals and using Proposition 1.

If χ is an infinite cardinal, $\mathcal{A}\subseteq\mathcal{P}(\chi)$ is an almost disjoint family in χ if $\forall x\in\mathcal{A}\,\forall y\in\mathcal{A}(x\neq y\Rightarrow|x|=\chi\ \&\ |x\cap y|<\chi)$.

Proposition 4. *Let $\chi\geq\omega$ be a regular cardinal. There exists a maximal almost disjoint family $\mathcal{B}\subset\mathcal{P}(\chi)$ of cardinality $\geq\chi^{+}$.*

Proof. [Ku].

We will prove now the main theorem

Theorem 5. *Let F be a free Boolean algebra and $A\subset F$ such that $|A|=\chi$, $cf\,\chi>\omega$. There exists a $B\subset A$ such that $|B|=\chi$ and B is an independent set.*

Proof. Let $C=\{c_\alpha\mid\alpha<\chi\}$ be the set of generators of F which occur in A. If $\varphi(x_0,\dots,x_{n-1})$ is a canonical form such that $a=\varphi[C_{\alpha(0)},\dots,C_{\alpha(n-1)}]$ for a strictly increasing function $\alpha\in^n\chi$, we will say that $\varphi[C_{\alpha(0)},\dots,C_{\alpha(n-1)}]$ is the canonical representation for a, $\varphi(x_0,\dots,x_{n-1})$ the canonical form for a and $\alpha=\alpha_a$ the canonical choice for a. For any given $a\in A$ they do exist and they are obviously unique. Since there are countably many different canonical forms and $\mathrm{cf}\,\chi>\omega$, there exists a $B\subset A$ such that $|B|=\chi$ and all the elements from B have the same canonical form e.g. $\varphi(x_0,\dots,x_{n-1})$. For the family $B=\{\alpha_a\mid a\subset B\}$ there exists, according to Proposition 3, a subset of the same cardinality χ which is a Δ-system. Without loss of generality we can suppose that it is the whole of B and there exists an $m<n$ such that the first m elements are in the root i.e. $\forall a\in B\,\forall b\in B\,(\alpha_a\uparrow m=\alpha_b\uparrow m\ \&\ \forall i,j\geq m(\alpha_a(i)\neq\alpha_b(j)))$. We claim that B is an independent family. Let $a_1,\dots,a_l,\ b_1,\dots,b_s\in D$. We will prove that $a_1\wedge\cdots\wedge a_l\wedge b_1'\wedge\cdots\wedge b_s'\neq 0$.

$$a_j=\vee\{\vee\{C^{f(i)}_{\alpha j(i)}\mid 1\leq j\leq l\}\mid f\in S_\varphi\},\quad b_k'=\vee\{\wedge\{C^{g(i)}_{\beta k(i)}\mid 1\leq k\leq S\}\mid g\in S_\varphi'\}$$

According to Lemma there exists $f\in S_\varphi$ and a $g\in S_\varphi'$ such that $f\uparrow m=g\uparrow m$. Then,

$$\wedge\{a_j\mid 1\leq j\leq l\}\wedge\wedge\{b_k'\mid 1\leq k\leq s\}\geq\wedge\{C^{f(i)}_{\alpha j(i)}\mid 0\leq i<n,\ 1\leq j\leq b\}$$

$$\wedge\{C^{g(i)}_{\beta k(i)}\mid 0\leq i\leq n,\ 1\leq k\leq s\}\geq\wedge\{C^{f(i)}_{\alpha j(i)}\mid 0\leq i\leq m\}$$

$$\wedge\{C^{f(i)}_{\alpha j(i)}\mid 1\leq j\leq l,\ m\leq i<n\}\wedge\wedge\{C^{g(i)}_{\beta k(i)}\mid 1\leq k\leq s,\ m\leq i<n\}$$

which is nonempty by the definition of independent set.

Corollary 6. *Let F be a free Boolean algebra and A a subalgebra of F such that $|A| = \chi$, $cf\chi > \omega$. A contains a free Boolean algebra of the same cardinality.*

Corollary 7. *$\mathcal{P}(\omega)$ can not be embedded in a free Boolean algebra.*

Proof. Suppose, to the contrary that $\mathcal{P}(\omega)$ can be embedded in a free Boolean algebra. According to Proposition 4 we can find a subfamily $\mathcal{A}$ of $\mathcal{P}(\omega)$ such that $|\mathcal{A}| = \omega_1$ $\forall a, b \in \mathcal{A}\,(a \neq b \Rightarrow |a| = \omega \;\&\; |a \cap b| < \omega)$. Then, according to Theorem 5, we can find an independent subfamily $\mathcal{B} \subset \mathcal{A}$ such that $|B| = \omega_1$. We will construct now a sequence $B_n \in \mathcal{B}$, $n \in \omega$ in the following way:

Let $B_0\, B_1$ be any two different elements from $\mathcal{B}$. We choose B_2 so that $|B_0 \cap B_1 \cap B_2| < |B_0 \cap B_1|$, if such an element exists, and so on.

Since $B_0 \cap B_1$ is finite, there exists an $n \in \omega$ at which we must stop our construction i.e. $B_0 \cap \cdots \cap B_{n-1} \cap B = B_0 \cap \cdots \cap B_{n-1}$ for every $B \in \mathcal{B}$. Choose $B_n \in \mathcal{B} \backslash \{B_0, \ldots, B_{n-1}\}$. But then we have that $B_0 \cap \cdots \cap B_{n-1} \cap B_n'$ is empty which contradicts our assumption that $\mathcal{B}$ is an independent family.

Corollary 8. *An interval algebra of cardinality χ, $cf\chi > \omega$ can not be embedded into a free Boolean algebra.*

Proof. A direct consequence of Theorem 5 and Theorem 5.5 from [Ru].

Using the Stone duality we will get topological duals of the last three results.

Corollary 9. *Let X be a topological space of weight χ, $cf\chi > \omega$. If X is a continuous image of a Cantor space, then the Cantor space $\mathcal{D} = \{0,1\}^\chi$ is a continuous image of X.*

Corrolary 10. *$\beta\omega$, the Stone-Čech compactification of the discrete space ω, is not a continuous image of any Cantor space.*

Corrolary 11. *An interval topological space of weight χ, $cf\chi > \omega$, is not a continuous image of any Cantor space.*

Remark. Special case of Theorem 5 for $\chi = \omega_1$ was proved in [Ef] and [Mo].

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