PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 40 (54), 1988, pp. 9–18

FORMULAS OF THE GENERAL SOLUTIONS OF BOOLEAN EQUATIONS

Dragić Banković

Abstract. We explicitly give various formulas of the general solutions of Boolean equations in n unknows. The method presented in the paper is based on a Prešić's idea of the solving function from [4], but we have it here in more general form. We build the cycle using the sequence $i_1, i_2, \dots, i_\nu$ ($\nu = 2^n$) where $\{i_1, i_2, \dots, i_\nu\} = \{0, 1, 2, \dots, \nu - 1\}$. We can chose the sequence so that we obtain the formulas of the general solution in the triangular form. Specially, when $i_1 = 2^n - 1$, we have the reproductive solutions. This paper enables one to make the program (we wrote it in FORTRAN IV) for digital computer which gives the formulas of the general solutions of Boolean equations, where the number of unknowns can be large. The limitation results only from the number of the elements of the sequence $i_1, i_2, \dots, i_\nu$ i.e. of the memory of the computer.

Let $X_n = (x_1, \dots, x_n) \in B^n$ and $T_n = (t_1, \dots, t_n) \in B^n$, where $(B, \cup, \cdot, ' 0, 1)$ is Boolean algebra.

Definition 1. Let $f : B^n \to B$ be a Boolean function. The system $\psi = (\psi_1, \dots, \psi_n)$ of Boolean functions $\psi_1, \dots, \psi_n : B^n \to B$ is a general solution of the consistent Boolean equation $f(X_n) = 0$ if and only if

$$(1) \qquad (\forall T_n) f(\psi(T_n)) = 0 \wedge (\forall X_n)(f(X_n) = 0 \Rightarrow (\exists T_n)(X_n = \psi(T_n)).$$

The system $\psi = (\psi_1, \dots, \psi_n)$ is a general reproductive solution of $f(X_n) = 0$ if and only if

$$(2) \qquad (\forall T_n) f(\psi(T_n)) = 0 \wedge (\forall X_n)(f(X_n) = 0 \Rightarrow x_n = \psi(X_n)).$$

Definition 2. [1]. The Horn formulas over language L are defined as follows

– the elementary Horn formulas are defined as the atomic formulas of L and the formulas of the form $F_1 \wedge \cdots \wedge F_k \Rightarrow G$, where $F_1, \dots, F_k$, G are atomic,

– every Horn formula is built form elementary Horn formulas by use of $\wedge, \forall, \exists$.

THEOREM 1. (Vaught) *Let H be a Horn sentence in language L_B of Boolean algebras. If $B_2 \models H$ then $B \models H$.*

AMS Subject Classification (1980): Primary 03G05

Let $i \in \{0,1,2,\dots,\nu-1\}$, where $\nu = 2^n$. Then $i(1),\dots,i(n)$ are the binary digits of the number i in binary expansion and $\vec{i} = (i(1),\dots,i(n))$. Let $Y = (\vec{y_o}, \vec{y_1}, \vec{y_2}, \dots, \vec{y_{\nu-1}})$ and

$$Z_{nxp} = \begin{pmatrix} z_{10} & z_{11} & z_{12} & \dots & z_{1,\ \nu-1} \\ z_{20} & z_{21} & z_{22} & \dots & z_{2,\ \nu-1} \\ \vdots & & & & \\ z_{n0} & z_{n1} & z_{n2} & \dots & z_{n,\ \nu-1} \end{pmatrix}$$

where we write $z_{k,i}$ instead of $z_{k,\vec{i}}$.

Corollary. *Let $f(X_n) = \cup_{A_n} y_{A_n} X_n^{A_n}$, $\varphi_k(x_n) = \cup_{A_n} z_{k,A_n} X^{A_n}$ $(k = 1,\dots,n)$ and $\varphi = (\varphi_1,\dots,\varphi_n)$.*

The formula

(3) $\quad (\forall Y)(\forall Z_{nxp})((\forall T_n) f(\varphi(T_n)) = 0 \wedge (\forall X_n)(\exists T_n)(f(X) = 0 \Rightarrow X_n = \varphi(T_n))$

holds in all Boolean algebras if and only if it holds in the Boolean algebra B_2.

Comment. In other words, if we have the "universal" formulas of the general solutions of Boolean equations in the Boolean algebra B_2 (these formulas are $x_i = \varphi_i(T_n)$ $(i = 1,\dots,n)$ then the same formulas are the "universal" formulas of the general solutions of Boolean equations in an arbitrary Boolean algebra B.

Proof. In accordance with Vaught's theorem the proof follows from the fact that (3) is a Horn sentence.

Obviously a similar corollary holds for the reproductive solutions.

Definition 2. Let $i \in \{0,1,2,\dots,\nu-1\}$. Then $T_n * i = (t_1^{i(1)},\dots,t_n^{i(n)})$. When $k < n$ then $T_k * i = (t_1^{i(1)},\dots,t_k^{i(k)})$ where $i(1),\dots,i(k)$ are the first k components of the vector $\vec{i} = (i(1),\dots,i(n))$.

Example 1. For $s = 4$ we have $T_4 * 3 = (t_1^0, t_2^0, t_3^1, t_n^1) = (\acute{t}_1, \acute{t}_2, t_3, t_4)$, because $(3)_{10} = (0011)_2$.

Lemma 1. *Let $f : B^n \to B$ be a Boolean function. Then*

$$f(T_n * i) = \cup_{A_n} f(A_n * i) T_n^{A_n} \ \ (i = 0,1,2,\dots,2^n - 1).$$

Proof. Since the last formula is an atomic formula it is sufficient to prove it in B_2. The proof in B_2 is obvious.

Lemma 2. *Let w be a natural number and $c_1, c_2, \dots, c_{2w+1} \in B$. Then*

(4) $\quad \acute{c}_1 \cup c_1\acute{c}_2 \cup c_1c_2\acute{c}_3 \cup \dots \cup c_1c_2c_3,\dots,c_{w-1}\acute{c}_w = \acute{c}_1 \cup \acute{c}_2 \cup \dots \cup \acute{c}_w$

Proof. Using the equality $x' \cup xy = x' \cup y$.

Definition 3. (i) $T_{n,k} = (t_{k+1},\dots,t_n) \quad (k \in \{1,\dots,n\})$

(ii) If $R = (p_1, \dots, p_r) \in B^r$ and $Q = (q_1, \dots, q_s) \in B^s$, where $r + s = n$, then $f(R, Q) = f(p_1, \dots, p_r, q_1, \dots, q_s)$

(iii) $P(T_n * i) = f'(T_n * i) \prod_{j=i+1}^{2n-1} f(T_n * j)$

We assume that $\prod_{i=m_1}^{m_2} = 1$ when $m_1 > m_2$.

(iv) $\prod_{A_{n,k}} f(T_k * p, A_{n,k})$ is a product over all $A_{n,k} = (a_{k+1}, \dots, a_n) \in \{0,1\}^{n-k}$ and we assume that $\prod_{An,n} f(T_k * p, A_{n,n}) = f(A_n * p)$.

LEMMA 3. *Let $f : B^n \to B$ be a Boolean function. Then*

$$\prod_{q=0}^{2n-k-1} f(T_k * p, T_{n,k} * q) = \prod_{A_{n,k}} f(T_k * p, A_{n,k}) \tag{5}$$

Proof. Since (5) is an atomic formula it is sufficient to prove it in B_2. The proof follows from the fact that

$$(\forall T_{n,k} \in \{0,1\}^{n-k})\{T_{n,k} * 0, T_{n,k} * 1, \dots, T_{n,k} * (2^{n-k} - 1)\} = \{0,1\}^{n-k}.$$

THEOREM 3. *Let $f : B^n \to B$ be a Boolean function and assume that the equation $f(X_n) = 0$ is consistent. Let $i_1, \dots, i_\nu$ $(\nu = 2^n)$ be a sequence of natural numbers such that $\{i_1, \dots, i_\nu\} = \{0, 1, 2, \dots, \nu - 1\}$. Then*

(a)
$$\begin{aligned}
X_n = {} & f'(T_n * i_1)(T_n * i_1) \cup f(T_n * i_1) f'(T_n * i_2)(T_n * i_2) \cup \dots \\
& \cup f(T_n * i_1) f(T_n * i_2) \dots f(T_n * i_{\nu-2}) f'(T_n * i_{\nu-1})(T_n * i_{\nu-1}) \\
& \cup f(T_n * i_1) f(T_n * i_2) \dots f(T_n * i_{\nu-2}) f(T_n * i_{\nu-1}) f'(T_n * i_\nu)(T_n * i_\nu)
\end{aligned} \tag{6}$$

or in scalar form

$$\begin{aligned}
x_k = {} & \cup_{A_n} (a_k^{i_1(k)} f'(A_n * i_1) \cup a_k^{i_2(k)} f(A_n * i_1) f'(A_n * i_2) \cup \dots \\
& \cup_{a_k}^{i_{\nu-1}(k)} f(A_n * i_1) f(A_n * i_2) \dots f(A_n * i_{\nu-2}) f'(A_n * i_{\nu-1}) \\
& \cup_{a_k}^{i_\nu(k)} f(A_n * i_1) f(A_n * i_2) \dots f(A_n * i_{\nu-2}) f(A_n * i_{\nu-1}) f'(A_n * i_\nu)) T_n^{A_n} \\
& (k = 1, \dots, n)
\end{aligned} \tag{7}$$

is a general solution of the equation $f(X_n) = 0$.

(b) *If $i_1 = 2^n - 1$ then the solution* (6) *is reproductive*

(c) *For every $e \in \{0, 1, 2, \dots, \nu - 1\}$ formulas* (6) *define a general solution in triangular form if $(i_1, \dots, i_p) = (\vec{e} * (2^n - 1),\ \vec{e} * (2^n - 2), \dots, \vec{e} * 1, e * \vec{0})$ and the scalar form of this solution is*

$$x_k = \cup_{a_k = e(k)\, A_k} (\cup_{i \in E} \prod_{j=i+1}^{2k-1} \prod_{A_{n,k}} f((A_n * e) * j, A_{n,k}))$$

$$(\prod_{A_{n,k}} f((A_k * e) * i, A_{n,k}))' (T_k * e)^{A_k} \tag{8}$$

$$\cup\cup'_{a_k\neq e(k)\,A_k}(\cup_{i\in F}\prod_{j=i+1}^{2k-2}\prod_{A_{n,k}} f(A_k*e)*j,A_{n,k}))$$
$$(\prod_{A_{n,k}} f((A_n*e)*i,A_{n,k}))'(T_k*e)^{A_k}$$

where

$$E=\{i|i\in\{0,1,2,\ldots,2^{k-1}\}\wedge i(k)=e(k)\},$$
$$F=\{i|i\in\{0,1,2,\ldots,2^{k-1}\}\wedge i(k)\neq e(k)\}$$

and $T_k*e=(t_1^{e(1)},\ldots,t_k^{e(k)})$. $\cup_{a_k=e(k)\,A_k}$ *means union over all* $A_k=(a_1,\ldots,a_k)\in\{0,1\}^k$ *such that* $a_k=e(k)$ *and* $\cup_{a_k\neq e(k)\,A_k}$ *means union over all* $A_k=(a_1,\ldots,a_k)\in\{0,1\}^k$ *such that* $a_k\neq e(k)$.

Comment. If we define the scalar product of two vectors $U=(u_1,\ldots,u_n)\in B^n$ and $V=(\nu,\ldots,\nu_n)\in B^n$ as $U\circ V=u_1\nu_1\cup\cdots\cup u_n\nu_n$ we can write formula (6) in the form

$$x_k=\cup_{A_n}(K(A_n)\circ F(A_n))T_n^{A_n}$$

where $K(A_n)=(a_k^{i_1(k)},a_k^{i_2(k)},\ldots,a_k^{i_\nu(k)})$ and

$$\begin{aligned}F(A_n)=(&f'(A_n*i_1),\ f(A_n*i_1)f'(A_n*i_2),\ldots,\\&f(A_n*i_1)f(A_n*i_2)\ldots f'(A_n*i_{\nu-1}),\\&f(A_n*i_1)f(A_n*i_2)\ldots f(A_n*i_{\nu-1})f'(A_n*i_\nu))\end{aligned}$$

Proof. In accordance with Corollary it is sufficient to prove the theorem in B_2. Let T_n*i_α be the first element of the sequence $T_n*i_1,\ldots,T_n*i_\nu$, satisfying equation $f(X_n)=0$. There exists such an element, because

$$(\forall T_n\in\{0,1\}^n)\{T_n*i_1,\ldots,T_n*i_\nu\}=\{0,1\}^n$$

and equation $f(X_n)=0$ is consistent. In this case formula (6) gives $X_n=T_n*i_\alpha$ i.e. (6) satisfies equation $f(X_n)=0$. Let $Y_n=(y_1,\ldots,y_n)\in B^n$ be a particular solution of equation $f(X_n)=0$ i.e. $f(Y_n)=0$. Let us prove that there exists $T_n\in B^n$ such that formula (6) gives $X_n=Y_n$. Putting $T_n=y_n*i_1$ the first of the unions of the right hand side of (6) becomes $f'((Y_n*i_1)*i_1)((Y_n*i_1)*i_1)$ i.e. $f'(Y_n)Y_n$ i.e. Y_n. The other elements of the union are equal to 0, because these elements contain $f(Y_n)$.

Applying Lemma 1 to formula (6) we have

$$\begin{aligned}X_n=&(\cup_{A_n}f'(A_n*i_1)T_n^{A_n})(T_n*i_1)\cup(\cup_{A_n}f(A_n*i_1)T_n^{A_n})(\cup_{A_n}f'(A_n*i_2)T_n^{A_n})\\&(T_n*i_2)\cup\ldots\cup(\cup_{A_n}f(A_n*i_1)T_n^{A_n})(\cup_{A_n}f(A_n*i_2)T_n^{A_n})\ldots\\&(\cup_{A_n}f'(A_n*i_{\nu-1})T_n^{A_n})(T_n*i_{\nu-1})\cup(\cup_{A_n}f(A_n*i_1)T_n^{A_n})(\cup_{A_n}f(A_n*i_2)T_n^{A_n})\ldots\\&(\cup_{A_n}f'(A_n*i_\nu)T_n^{A_n})(T_n*i_\nu)\end{aligned}$$

and denoting the k-th component of $T_n * i_m$ by $(T_n * i_m)_k$ from

$$T_n^{A_n}(T * i_m)_k = t_1^{a_1} \cdots t_n^{a_n} t_k^{i_m(k)} = a_k^{i_m(k)} T_n^{A_n}$$

we get the scalar form i.e. (7).

(b) Let $i_1 = 2^n - 1$ and $f(Y_n) = 0$ putting $T_n = Y_n$ the first element of the union of the right hand side of (6) becomes $f'(Y_n)Y_n$ i.e. Y_n because $Y_n * (2^n - 1) = Y_n$. The other elements of the union are 0.

(c) Formula (6) can be written as

$$X_n = \cup_{m=0}^{2n-1} P(T_n * m)(T_n * m)$$

$$X_n = \cup_{m=0}^{2n-1} P((T_n * e) * m)((T_n * e) * m) \qquad (e \in \{0, 1, 2, \ldots, p-1\})$$

or in scalar form

$$x_k = \cup_{m=0}^{2n-1} P((T_n * e) * m)((T_n * e) * m)_k$$

define also a general solution of equation $f(X_n) = 0$. Let $S_n = T_n * e$. Obviously

$$\begin{aligned}
&(S_n * (2^n - 1),\ S_n * (2^n - 2), \ldots, S_n * 1,\ S_n * 0) = \\
&= ((S_k * (2^k - 1),\ S_{n,k} * (2^{n-k} - 1)),\ (S_k * (2^k - 1),\ S_{n,k} * (2^{n-k} - 2), \ldots \\
&\ldots,\ (S_k * (2^k - 1),\ S_{n,k} * 0),\ (S_k * (2^k - 2),\ S_{n,k} * (2^{n-k} - 1)),\ (S_k * (2^k - 2), \\
&S_{n,k} * (2^{n-k} - 2), \ldots, (S_k * (2^k - 1),\ S_{n,k} * 0), \ldots \\
&\vdots \\
&(S_k * 0,\ S_{n,k} * (2^{n-k} - 1)),\ (S_k * 0,\ S_{n,k} * (2^{n-k} - 2)), \ldots, (S_k * 0,\ S_{n,k} * 0))
\end{aligned}$$

and we have

$$x_k = \cup_{i=0}^{2k-1} \cup_{j=0}^{2n-k-1} P(S_k * i,\ S_{n,k} * j)(S_k * i,\ S_{n,k} * j)_k \ \ (k = 1, \ldots, n).$$

Since

$$(S_k * i,\ S_{n,k} * j)_k = (t_k^{e(k)})^{i(k)} = \begin{cases} t_k & \text{for } i(k) = e(k) \\ t_k' & \text{for } i(k) \neq e(k) \end{cases}$$

we get

$$x_k = (\cup_{i \in E} \cup_{j=0}^{2n-k-1} P(S_k * i,\ S_{n,k} * j) t_k \cup (\cup_{i \in F} \cup_{j=0}^{2n-k-1} P(S_k * i,\ S_{n,k} * j) t_k'.$$

Notice

$$F = \begin{cases} \{0, 2, 4, \ldots, 2^k - 2\} & \text{for } e(k) = 1 \\ \{1, 3, 5, \ldots, 2^k - 1\} & \text{for } e(k) = 0 \end{cases} \quad E = \begin{cases} \{1, 3, 5, \ldots, 2^k - 1\} & \text{for } e(k) = 1 \\ \{0, 2, 4, \ldots, 2^k - 2\} & \text{for } e(k) = 0. \end{cases}$$

Further

$$\cup_{j=0}^{2n-k-1} P(S_k * i,\ S_{n,k} * j) =$$
$$=P(S_k * i,\ S_{n,k} * 0) \cup P(S_k * i,\ S_{n,k} * 1) \cup \cdots \cup P(S_k * i,\ S_{n,k} * (2^{n-k} - 1))$$
$$=(\prod_{p=i+1}^{2k-1} \prod_{q=0}^{2n-k-1} f(S_k * p,\ S_{n,k} * q))(f'(S_k * i,\ S_{n,k} * (2^{n-k} - 1)) \cup$$
$$\cup f(S_k * i,\ S_{n,k} * (2^{n-k} - 1)) f'(S_k * i,\ S_{n,k} * (2^{n-k} - 2)) \cup \cdots$$
$$\cup f(S_k * i,\ S_{n,k} * (2^{n-k} - 1)) f(S_k * i,\ S_{n,k} * (2^{n-k} - 2)) \cdots f'(S_k * i,\ S_{n,k} * 0)) =$$
$$=(\prod_{p=i+1}^{2k-1} \prod_{A_{n,k}} f(S_k * p,\ A_{n,k}))(f'(S_k * i,\ S_{n,k} * (2^{n-k} - 1)) \cup$$
$$\cup f'(S_k * i,\ S_{n,k} * (2^{n-k} - 2)) \cup \cdots \cup f'(S_k * i,\ S_{n,k} * 0))$$

(using Lemma 3 and (4))

$$=(\prod_{p=i+1}^{2k-1} \prod_{A_{n,k}} f(S_k * p,\ A_{n,k})) \cdot (\prod_{q=0}^{2n-k} f(S_k * i,\ S_{n,k} * q))'$$
$$=(\prod_{p=i+1}^{2k-1} {}_{A_{n,k}} f(S_k * p,\ A_{n,k}))(\prod_{A_{n,k}} f(S_k * i,\ A_{n,k}))'$$

(using Lemma 3).

In the following steps, we use Lemma 1, the distributive law and the well known equalities

$$\cup\, c_n a c_n x_n^{c_n} (\cup c_n b c_n x_n^{c_n}) = \cup c_n a c_n X_n^{c_n}$$
$$(\cup c_n a c_n X_n^{c_n})' = \cup c_n a' c_n x_n^{c_n})$$

$$x_k = (\ \cup_{i \in E} (\prod_{p=i+1}^{2k-1} \prod_{A_{n,k}} f(S_k * p,\ A_{n,k})))(\prod_{A_{n,k}} f(S_k * i,\ A_{n,k}))' t_k \cup$$
$$\cup \cup_{i \in F} (\prod_{p=i+1}^{2k-1} \prod_{A_{n,k}} f(S_k * p,\ A_{n,k}))(\prod_{A_{n,k}} f(S_k * i,\ A_{n,k}))' t_k'.$$

Let $B_k = (a_1^{e(1)}, \ldots, a_k^{e(k)})$ i.e. $B_k = A_k * e$. Then

$$x_k = (\ \cup_{i \in E} (\prod_{p=i+1}^{2k-1} \prod_{A_{n,k}} (\cup_{A_k} f(B_k * p,\ A_{n,k}) S_k^{A_k})(\prod_{A_{n,k}} (\cup_{A_k} f(B_k * i,$$
$$A_{n,k}) S_k^{A_k}))' t_k \cup (\cup_{i \in F} (\prod_{p=i+1}^{2k-1} \prod_{A_{n,k}} (\cup_{A_k} f(B_k * p,\ A_{n,k}) S_k^{A_k})))(\prod_{A_{n,k}}$$
$$(\ \cup_{A_k}\ f(B_k * i,\ A_{n,k}) S_k^{A_k}))' t_k = \{\cup_{A_k} \cup_{i \in E} (\prod_{p=i+1}^{2k-1} \prod_{A_{n,k}} f(B_k * p,\ A_{n,k})\}$$

$$\{\cup_{A_k}(\prod_{A_{n,k}} f(B_k*i,\ A_{n,k})S_k^{A_k})'\}t_k \cup \{\cup_{A_k}\cup_{i\in F}\prod_{p=i+1}^{2k-1}\prod_{A_{n,k}} f(B_k*p,\ A_{n,k})\}$$

$$\{\cup_{A_k}(\prod_{A_{n,k}} f(B_k*i,\ A_{n,k})S_k^{A_k})'\}t_k' = \{\cup_{A_k}(\cup_{i\in E}\prod_{p=i+1}^{2k-1}\prod_{A_{n,k}} f(B_k*p,\ A_{n,k}))$$

$$(\prod_{A_{n,k}} f(B_k*i,\ A_{n,k}))'S_k^{A_k}\}t_k \cup \{\cup_{A_k}(\cup_{i\in F}\prod_{p=i+1}^{2k-1}\prod_{A_{n,k}} f(B_k*p,\ A_{n,k}))$$

$$(\prod_{A_{n,k}} f(B_k*i,\ A_{n,k}))'S_k^{A_k}\}t_k' = \{\cup_{A_k}(\cup_{a(k)=e(k),i\in E}\prod_{p=i+1}^{2k-1}\prod_{A_{n,k}} f(B_k*p,\ A_{n,k}))$$

$$(\prod_{A_{n,k}} f(B_k*i,\ A_{n,k}))'(T_k*e)^{A_k}\}t_k \cup \{\cup_{A_k}(\cup_{a(k)\neq e(k),i\in F}\prod_{p=i+1}^{2k-1}\prod_{A_{n,k}} f(B_k*p,\ A_{n,k}))$$

$$(\prod_{A_{n,k}} f(B_k*i,\ A_{n,k}))'(T_k*e)^{A_k}\}t_k'$$

i.e. (8).

Example 2. The program gives the following result for $n = 3$ and $(i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8) = (6, 1, 0, 7, 3, 5, 4, 2)$.

$X1 =$

$T3{*}{*}000$

pr 6, 1,
pr 7, 1, ∗ ∗ ∗,
pr 4, 1, ∗ ∗ ∗, ∗ ∗ ∗, 0,
1, ∗ ∗ ∗, ∗ ∗ ∗, 0, ∗ ∗ ∗, 2, 3,

$T3{*}{*}001$

pr 7, 0,
pr 6, 0, ∗ ∗ ∗,
pr 5, 0, ∗ ∗ ∗, ∗ ∗ ∗, 1,
0, ∗ ∗ ∗, ∗ ∗ ∗, 1, ∗ ∗ ∗, 3, 2,

$T3{*}{*}010$

pr 4, 3,
pr 5, 3, ∗ ∗ ∗,
pr 6, 3, ∗ ∗ ∗, ∗ ∗ ∗, 2,
3, ∗ ∗ ∗, ∗ ∗ ∗, 2, ∗ ∗ ∗, 0, 1,

$T3{*}{*}011$

$pr\quad 5,\quad 2,$
$pr\quad 4,\quad 2,***,$
$pr\quad 7,\quad 2,***,***,\quad 3,$
$2,***,***,\quad 3,***,\quad 1,\quad 0,$

$T3{*}{*}100$

$pr\quad 5,$
$pr\quad 4,***,\quad 2,\quad 3,$
$pr\quad 6,***,\quad 2,\quad 3,***,\quad 0,$
$pr\quad 7,***,\quad 2,\quad 3,***,\quad 0,***,$

$T3{*}{*}101$

$pr\quad 4,$
$pr\quad 5,***,\quad 3,\quad 2,$
$pr\quad 7,***,\quad 3,\quad 2,***,\quad 1,$
$pr\quad 6,***,\quad 3,\quad 2,***,\quad 1,***,$

$T3{*}{*}110$

$pr\quad 7,$
$pr\quad 6,***,\quad 0,\quad 1,$
$pr\quad 4,***,\quad 0,\quad 1,***,\quad 2,$
$pr\quad 5,***,\quad 0,\quad 1,***,\quad 2,***,$

$T3{*}{*}111$

$pr\quad 6,$
$pr\quad 7,***,\quad 1,\quad 0,$
$pr\quad 5,***,\quad 1,\quad 0,***,\quad 3,$
$pr\quad 4,***,\quad 1,\quad 0,***,\quad 3,***,$

$X_2 =$

$T3{*}{*}000$

$pr\quad 6,\quad 1,$
$pr\quad 7,\quad 1,***,$
$pr\quad 2,\quad 1,***,***,\quad 0,\quad 4,$
$pr\quad 3,\quad 1,***,***,\quad 0,\quad 4,***,$

$T3{*}{*}001$

$pr\quad 7,\quad 0,$
$pr\quad 6,\quad 0,***,$
$pr\quad 3,\quad 0,***,***,\quad 1,\quad 5,$
$pr\quad 2,\quad 0,***,***,\quad 1,\quad 5,***,$

$T3{*}{*}010$

$pr\ \ 3,$
$pr\ \ 2,***,\ \ 4,\ \ 5,$
$pr\ \ 6,***,\ \ 4,\ \ 5,***,$
$***,\ \ 4,\ \ 5,***,***,\ \ 0,\ \ 1,$

$T3{*}{*}011$

$pr\ \ 2,$
$pr\ \ 3,***,\ \ 5,\ \ 4,$
$pr\ \ 7,***,\ \ 5,\ \ 4,***,$
$***,\ \ 4,\ \ 5,***,***,\ \ 1,\ \ 0,$

$T3{*}{*}100$

$pr\ \ 2,\ \ 5,$
$pr\ \ 3,\ \ 5,***,$
$pr\ \ 6,\ \ 5,***,***,\ \ 4,\ \ 0,$
$pr\ \ 7,\ \ 5,***,***,\ \ 4,\ \ 0,***,$

$T3{*}{*}101$

$pr\ \ 3,\ \ 4,$
$pr\ \ 2,\ \ 4,***,$
$pr\ \ 7,\ \ 4,***,***,\ \ 5,\ \ 1,$
$pr\ \ 6,\ \ 4,***,***,\ \ 5,\ \ 1,***,$

$T3{*}{*}110$

$pr\ \ 7,$
$pr\ \ 6,***,\ \ 0,\ \ 1,$
$pr\ \ 2,***,\ \ 0,\ \ 1,***,$
$***,\ \ 0,\ \ 1,***,***,\ \ 4,\ \ 5,$

$T3{*}{*}111$

$pr\ \ 6,$
$pr\ \ 7,***,\ \ 1,\ \ 0,$
$pr\ \ 3,***,\ \ 1,\ \ 0,***,$
$***,\ \ 1,\ \ 0,***,***,\ \ 4,\ \ 5,$

$X_3 =$

$T3{*}{*}000$

$pr\ \ 1,$
$pr\ \ 7,***,\ \ 6,$
$pr\ \ 3,***,\ \ 6,***,\ \ 0,\ \ 5,\ \ 2,$
$***,\ \ 6,***,\ \ 0,\ \ 5,\ \ 2,***,$

$T3{*}{*}001$

$pr\quad 7,\quad 0,$
$pr\quad 1,\quad 0,*{*}*,\quad 6,$
$pr\quad 5,\quad 0,*{*}*,\quad 6,*{*}*,$
$pr\quad 3,\quad 0,*{*}*,\quad 6,*{*}*,*{*}*,$

$T3{*}{*}010$

$pr\quad 3,$
$pr\quad 5,*{*}*,\quad 4,$
$pr\quad 1,*{*}*,\quad 4,*{*}*,\quad 2,\quad 6,\quad 0,$
$*{*}*,\quad 4,*{*}*,\quad 2,\quad 6,\quad 0,*{*}*,$

$T3{*}{*}011$

$pr\quad 5,\quad 2,$
$pr\quad 3,\quad 2,*{*}*,\quad 4,$
$pr\quad 7,\quad 2,*{*}*,\quad 4,*{*}*,$
$pr\quad 1,\quad 2,*{*}*,\quad 4,*{*}*,*{*}*,$

$T3{*}{*}100$

$pr\quad 5,$
$pr\quad 3,*{*}*,\quad 2,$
$pr\quad 7,*{*}*,\quad 2,*{*}*,\quad 4,\quad 0,\quad 6,$
$*{*}*,\quad 2,*{*}*,\quad 4,\quad 0,\quad 6,*{*}*,$

$T3{*}{*}101$

$pr\quad 3,\quad 4,$
$pr\quad 5,\quad 4,*{*}*,\quad 2,$
$pr\quad 1,\quad 4,*{*}*,\quad 2,*{*}*,$
$pr\quad 7,\quad 4,*{*}*,\quad 2,*{*}*,*{*}*,$

$T3{*}{*}110$

$pr\quad 7,$
$pr\quad 1,*{*}*,\quad 0,$
$pr\quad 5,*{*}*,\quad 0,*{*}*,\quad 6,\quad 2,\quad 4,$
$*{*}*,\quad 0,*{*}*,\quad 6,\quad 2,\quad 4,*{*}*,$

$T3{*}{*}111$

$pr\quad 1,\quad 6,$
$pr\quad 7,\quad 6,*{*}*,\quad 0,$
$pr\quad 3,\quad 6,*{*}*,\quad 0,*{*}*,$
$pr\quad 5,\quad 6,*{*}*,\quad 0,*{*}*,*{*}*,$

Instead of $T_n^{(a,\dots,a_n)}$ in (7) we print $T_n \exp a_1 \dots a_n$. The output after $T_n \exp a_1 \dots a_n$ means the union of the products where $f'(m_1)f(m_2)\dots f(m_r)$

is coded as pr $m_1, m_2, \ldots, m_r$ and m_s is the vector $(m_s(1), \ldots, m_s(n))$, where $m_s(1), \ldots, m_s(n)$ are the binary digits of the number m_s in binary expansion. For instance, pr $6, ***, 0, 1$ means $f'(6)f(0)f(1)$ i.e. $f'(1,1,0)f(0,0,0)f(0,0,1)$. The marks $***$ have no importance.

We conclude from Theorem 3 that we can get the formulas of $2^n!$ general solutions, the formulas of $(2^n-1)!$ general reproductive solutions and the formulas of 2^n general solutions in the triangular form.

The computer results suggest that for $(i_1, \ldots, i_\nu) = (2^n-1, 2^n-2, \ldots, 1, 0)$ the formulas (6) are the same as the formulas obtained by the method of successive eliminations.

REFERENCES

[1] C.C. Chang, H.J. Keisler, *Model Theory*, North-Holland, Amsterdam, 1973.

[2] F.M. Brown, S. Rudeanu, *Reccurent covers and Boolean equations*, Colloquia Math. Soc. János Bolyai **33** (Lattice Theory), Szeged (Hungary), 1980, (1983), 637–650.

[3] F.M. Brown, S. Rudeanu, *Triangular reproductive solutions of Boolean equations*, An. Univ. Craiova, Ser. Mat. Fiz.-Chim., **13** (1985), 18–23.

[4] Ž. Mijajlović, *Some remarks on Boolean terms-model theoretical approach*, Publ. Inst. Math. (Beograd), **21** (**35**) (1977), 135–140.

[5] Ž. Mijajlović, *Two remarks on Boolean algebras*, Algebraic Conference, Skoplje, 1980.

[6] S. Prešić, *Une méthode de résolution des équations dont toutes les solutions appartiennent à un ensemble fini donné.* C.R. Acad. Sci. Paris **272** (1989), 654–657.

[7] S. Rudeanu, *Boolean equations and their applications to the study of bridge circuits.* I. Bull. Math. Soc. Sci. Math. Phys. R.P. Roumaine **3** (**51**) (1991), 445–473.

[8] S. Rudeanu, *Boolean Functions and Equations*, North-Holland, Amsterdam, 1974.

Prirodno-matematički fakultet
34000 Kragujevac
Jugoslavija

(Received 23 06 1987)
(Revised 15 04 1988)