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NUMERICAL SOLUTION OF INITIAL AND SINGULARLY PERTURBED TWO-POINT BOUNDARY VALUE PROBLEMS USING ADAPTIVE SPLINE FUNCTION APPROXIMATION

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Abstract A special adaptive spline function which depends on arbitrary parameter q is constructed. The form of this spline is exponential and in a limiting case when $q \to 0$ it reduces to cubic spline. The schemes which have been applied to the initial and boundary value problems have been derived by using the adaptive spline function of special form and its smoothness properties. The approximative properties of adaptive spline function are illustrated by numerical experiments.

1. Introduction. An adaptive spline function to solve initial and boundary value problems of ordinary and partial differential equation was introduced in [**4**]. That spline was obtained as a solution of a singularly perturbed differential equation with the first derivative term. In a limiting case that adaptive spline function reduces to cubic spline. The numerical method based on that function gives better results than cubic spline or some other well known numerical methods (trapezodial for instance).

In [**5, 7**] it was proved that exponentially fitted cubic spline difference scheme has also better accuracy than common cubic spline.

In this paper, in the derivation of adaptive spline function, the form of singularly perturbed two-point boundary value problem lacking the first derivative is used.

The schemes were constructed in the way described in [**4**] but for the problem lacking the first derivative.

It is shown that the difference schemes, specially constructed using adaptive spline function (that gives the cubic spline in a limiting case), have beter results than cubic spline, even exponentially fitted cubic spline.

They are applicable to the initial value problems and for the singularly perturbed boundary value problems lacking the first derivative term. These schemes

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are less complex than those in [**4**], but they can achieve a uniform second order accuracy in a small parameter ε for the second problem mentioned above.

2. Derivation and Properties of the Adaptive Spline Function. We consider a mesh with nodal points x_j on the interval $[0,1]$ such that:

$$\Delta : 0 = x_0 < x_1 < \cdots < x_n = 1 \text{ where } h = x_j - x_{j-1} \text{ for } j = 0(1)n.$$

We denote a function $u(x_j)$ bu u_j, and $S_\Delta(x_{j-1}, q) = u_{j-1}$, $S_\Delta(x_j, q = u_j$ as interpalatory constraints. Define the adaptive spline function $S_\Delta(x,q)$ as a solution of differential equation:

$$-\varepsilon S^n_\Delta(x,q) + pS_\Delta(x,q) = (x - x_{j-1})h^{-1}(-\varepsilon M_j + pu_j) + \\ +(x_j - x)h^{-1}(-\varepsilon M_{i-1} + pu_{j-1}) \tag{1}$$

where $x_{j-1} \le x \le x_j, \varepsilon, p$ are constants, $S^n_\Delta(x_j,q) = M_j$ and $q = (p/\varepsilon)^{1/2}h$. This definition of adaptive spline differs from the definition given in [**4**] since there exists a term with the first derivative but the term with the function is missing, and in the above definition the term with the first derivative is lacking but there is the term with the function. In the following text under adaptive spline will be always unedrstood adaptive spline in the sense of our definition.

Solving (1) we obtain:

$$S_\Delta(x,q) = \varepsilon(p\,\mathrm{sh}g)^{-1}\{M_j \mathrm{sh}((p/\varepsilon)^{1/2}(x - x_{j-1})) - M_{j-1}\mathrm{sh}((p/\varepsilon)^{1/2}(x - x_j))\} + \\ +x(ph)^{-1}\{\varepsilon(M_j - M_{j-1}) + p(u_j - u_{j-1})\} + \\ +(ph)^{-1}\{\varepsilon(x_{j-1}M_j - x_jM_{j-1}) - p(x_{j-1}u_j - x_ju_{j-1})\}. \tag{2}$$

So we propose an adaptive spline function which depends on a arbitrary parameter to be chosen to suit a particular problem.

Introducting $z = (x - x_{j-1})h^{-1}$ and $1 - z = (x_j - x)h^{-1}$ we get

$$S_\Delta(z,q) = h^2q^{-2}(\mathrm{sh}\,q)^{-1}(M_j\mathrm{sh}(qz) + M_{j-1}\mathrm{sh}(q(1-z))) + \\ + h^2q^{-2}(-M_jz + M_{j-1}(1-z)) + u_jz + u_{j-1}(1-z). \tag{3}$$

In the limiting case when $q \to 0$, the representation (3) reduces to the well-known cubic spline [**1**]

$$S_\Delta = \frac{h^2}{6}M_jz = \frac{h^2}{6}M_{j-1}(1-z)^3 + z\Big(u_j - \frac{h^2}{6}M_j\Big) + (1-z)\Big(u_{j-1} - \frac{h^2}{6}M_{j-1}\Big),$$

i.e.

$$S_\Delta = M_{j-1}\frac{(x_j - x)^3}{6h} + M_j\frac{(x - x_{j-1})^3}{6h} + \Big(u_{j-1} - M_{j-1}\frac{h^2}{6}\Big)\frac{(x_j - x)}{h} + \\ + \Big(u_j - M_j\frac{h^2}{6}\Big)\frac{(x - x_{j-1})}{h}.$$

It can be seen from (1) that for $p = 0$ we obtain cubic spline, and for $\varepsilon = 0$ linear spline. Also, if $h/\sqrt{\varepsilon} \to 0$, we obtain cubic spline as $p \neq 0$.

As we can see from (2) the function $S_\Delta(x,q)$ belongs to the class $C^2[0,1]$, interpolates function $u(x)$ at the mesh points x_j, depends on a parameter q, and it reduces to polynomial cubic spline as $q \to 0$.

Letter it was recognized that adaptive spline is a tension spline when all tension parameters are equal, i.e. the case of uniform tension. All those splines have the base

$$S(x) \in \text{ span } \{1, x, e^{\varrho_i x}, e^{-\varrho_i x}\}, \ \ x_{i-1} < x < x_i$$

when ϱ_i is a tension parameter. When $\varrho_i \to 0$, cubic spline arise. We can say that cubic spline is an adaptive or tension spline with uniform tension to zero.

If $q \to \infty$, i.e. $\varepsilon \to 0$ then (3) leads to linear interpolant $S_\Delta(x) = u_j z + (1 - z)u_{j-1}$. The properties of interpolatory splines in tension and convergence analysis of the behaviour for large parameter is described in [**6**].

By using the condition of continuity of the first derivative of $S_\Delta(x,q)$ at the point x_j we get the folowing equation:

$$u_{j+1} - 2u_j + u_{j-1} = h^2 q^{-2}[M_{j-1}(1 - q(\operatorname{sh} q)^{-1} + 2M_j(-1 + q\operatorname{cth} q) + M_{j+1}(1 - q(\operatorname{sh} q)^{-1})] \tag{4}$$

We have some additional spline relations:

$$m_j = M_j\Big\{\frac{h}{q}\operatorname{cth} q - \frac{h}{q^2}\Big\} + M_{j-1}\Big\{-\frac{h}{q\operatorname{sh} q} + \frac{h}{q^2}\Big\} = \frac{1}{h}(u_j - u_{j-1}) \tag{5}$$

$$m_{j-1} = M_j\Big\{\frac{h}{q\operatorname{sh} q} - \frac{h}{q^2}\Big\} + M_{j-1}\Big\{-\frac{h}{q}\operatorname{cth} q + \frac{h}{q^2}\Big\} + \frac{1}{h}(u_j - u_{j-1}) \tag{6}$$

where m_j denotes the first derivative at the point x_j, i.e. $m_j = S'_\Delta(x_j, q)$.

The second derivative of A.S.F. can be expressed by means of the first one

$$M_j = D_s^{-1} h q^{-2}\{[m_j(1 - q\operatorname{cth} q)] - h^{-1}(u_j - u_{j-1})(1 - q\operatorname{cth} q) - m_{j-1}(1 - q(\operatorname{sh} q)^{-1} + +h^{-1}(1 - q(\operatorname{sh} q)^{-1} + h^{-1}(1 - q(\operatorname{sh} q)^{-1}(u_j - u_{j-1})\} \tag{7}$$

$$M_{j-1} = D_s^{-1} h q^{-2}\{m_{j-1}(1 - q\operatorname{cth} q - 1) - h^{-1}(q\operatorname{cth} q - 1)(u_j - u_{j-1}) - m_j(q(\operatorname{sh} q)^{-1} - 1 + h^{-1}(q(\operatorname{sh} q)^{-1} - 1)(u_j - u_{j-1})\} \tag{8}$$

where

$$D_s = (hq^{-1})^2 q\{(\operatorname{sh} q)^{-1} - \operatorname{cth} q\}\{q\operatorname{cth} q + q(\operatorname{sh} q)^{-1} - 2\}.$$

From (5) and (6) we obtain (4), (7) and (8) give

$$\frac{q}{h} th\frac{q}{2}(u_{j+1} - u_{j-1}) = m_{j+1}\Big(1 - \frac{q}{\operatorname{sh} q}\Big) + 2m_j(q\operatorname{cth} q - 1) + m_{j-1}\Big(1 - \frac{q}{\operatorname{sh} q}\Big). \tag{9}$$

Remark 1. When $q \to 0$, (9) becomes

$$(u_{j+1} - u_{j-1})2^{-1}h^{-1} = (u'_{j+1} + 4u'_j + u'_{j-1})6^{-1}$$

which is a well-known relation for the cubic spline.

Remark 2. When $q \to 0$, the relation (4) leads to the well-known result for polynomial cubic spline concerning the second derivatives (see [**11**]).

Using operator notation (see [**3**]), (4) can be written in the form

$$(E - 2I + E^{-1})u_j = (h^2/q^2)[E^{-1}(1 - q/\mathrm{sh}\, q) + 2(-1 + q\mathrm{cth}\, q) + E(1 - q/\mathrm{sh}\, q)]M_j,$$

where $Eu_j = u_{j+1},\ Iu_j = u_j$. Hence,

$$M_j = q^2/h^2(E - 2I + E^{-1}[E^{-1}(1 - q/\mathrm{sh}\, q) + 2(-1 + q\mathrm{cth}\, q) + E(1 - q/\mathrm{sh}\, q)]^{-1}u_j.$$

Operator E can be written as $E = e^{hD}$, where E and D are the shift and the differential operators respectively.

The same operator technique is applied to (9). Expanding (4) and (9) in powers of hd, we obtain:

$$m_j = u'_j = \frac{h^2}{6}\Big[1 - \frac{3}{q}\mathrm{cth}\frac{q}{2}\Big(1 - \frac{q}{\mathrm{sh}\ q}\Big)\Big]u_j^{'''} + \cdots \tag{10}$$

$$M_j = \Big(\frac{q}{2}\mathrm{cth}\frac{q}{2}\Big)u^{''} + \frac{h^2}{12}\Big[\frac{q}{2}\mathrm{cth}\frac{q}{2} - 3\mathrm{cth}^2\frac{q}{2}\Big(1 - \frac{q}{\mathrm{sh}\, q}\Big)\Big]u_j^{iv} + \cdots \tag{10.b}$$

Remark 3. When $q \to 0$ we obtain:

$$m_j = u'_j + \frac{h^2}{6}\Big(\frac{q^2}{12}\Big)u_j^{'''} + \cdots,\ M_j = u_j^{''} + \frac{h^2}{12}\Big(\frac{q^2}{20}u_j^{iv} + \cdots.$$

Differentiating (1) and the corresponding equation in $[x_j, x_{j+1}]$ and puting $x = x_j$ we obtain:

$$\begin{aligned} S_\Delta^{'''}(x_j^-, q) =& q^2h_{j-1}^{-2}m_j + h_{j-1}^{-1}(M_j - M_{j-1} + q^2h_{j-1}^{-2}(u_{j-1} - u_j)) \\ S_\Delta^{'''}(x_j^+, q) =& q^2h_j^{-2}m_j + h_j^{-1}(M_{j+1} - M_j + q^2h_j^{-2}(u_j - u_{j+1})) \end{aligned} \tag{11}$$

For higher derivatives we have the recursion formula:

$$S_\Delta^{(n)}(x_j^\pm, q) = q^2h^{-2}S_\Delta^{(n-1)}(x_j, q)\ \ n \geq 4 \tag{12}$$

Let $e(x) = u(x) - S_\Delta(x,q)$ be the interpolation error for the adaptive spline function approximation. Making use of the Taylor's expansion to equations (10-12) we can establish the error formula:

$$e(x_j+\vartheta h) = u(x_j+\vartheta h) - S_\Delta(x_j+\vartheta h, q) = \frac{\vartheta h^3}{1!}u_j^{'''}\left[-\frac{1}{6}+\frac{1}{2q}\mathrm{cth}\frac{q}{2}\Big(1-\frac{q}{\mathrm{sh}\, q}\Big)\right]+$$
$$+\cdots+\frac{(\vartheta h)^2}{2!}\left[u_j^{''}\Big(1-\frac{q}{2}\mathrm{cth}\frac{q}{2}\Big)-u_j^{iv}\frac{h^2}{12}\Big\{\frac{q}{2}\mathrm{cth}\frac{q}{2}-3\mathrm{cth}^2\frac{q}{2}\Big(1-\frac{q}{\mathrm{sh}\, q}\Big)\Big\}\right]+$$
$$+\frac{(\vartheta h)^3}{3!}\left[u_j^{'''}\Big(1-\frac{q}{1}\mathrm{cth}\frac{q}{2}\Big)+\frac{q^2}{4h}u_j^{''}+u_j^{iv}\frac{h^2}{12}\Big\{\frac{q}{2}\mathrm{cth}\frac{q}{2}-\right.$$
$$\left.-3\mathrm{cth}^2\frac{q}{2}\Big(1-\frac{q}{\mathrm{sh}\, q}\Big)-q^2h\Big\}+\cdots \qquad (13)$$

When $q \to 0$ the truncation error of this method is $o(h^4\varepsilon^{-1})$. When $\varepsilon = 1$, the error is $o(h^4)$.

In developed form $e(x_j+\vartheta h)$ has the form:

$$e(x_j+\vartheta h) = u_j^{''}\frac{\vartheta^2h^2q^2}{24}(\vartheta-1)+u_j^{'''}\frac{\vartheta h^3}{1!}\frac{q^2}{72}(1-\vartheta^2)+u_j^{iv}\cdot \qquad (14)$$

3. Applications of Adaptive Spline Function. The applications of the adaptive spline function relations to solve both the initial and singularly perturbed two-point boundary value problems will be analysed.

With (9) we associated a linear operator:

$$L[u(x_j),h] = \frac{q}{h}\mathrm{th}\frac{q}{2}(u(x_{j+1})-u(x_{j-1})(1-(q(\mathrm{sh}\, q)^{-1})-$$
$$-2u'(x_j)(q\mathrm{cth}\, q-1)-u'(x_{j-1}(1-q(\mathrm{sh}\, q)^{-1} \qquad (15)$$

Expanding each term on the right-hand side of (15), in Taylor's series at $x = x_j$ and collecting these terms we obtain:

$$L[u(x_j),h] =$$
$$= u_j^{'''}h^2\left[\frac{1}{3}q\mathrm{th}\frac{q}{2}-\Big(1-\frac{q}{\mathrm{sh}\, q}\Big)\right]+u_j^{v}h^4\left[\frac{q}{6}\mathrm{th}\frac{q}{2}-\frac{1}{12}\Big(1-\frac{q}{\mathrm{sh}\, q}\Big)\right]+\cdots \qquad (16)$$

We apply (9) to the test equation

$$u' = \lambda u; \qquad u(0) = 1; \qquad \lambda < 0 \qquad (17)$$

Since $m_j = \lambda u_j$ and $q = \lambda h$, the resulting difference scheme is:

$$u_{j+1}\{\mathrm{th}(q/2)-(1-q(\mathrm{sh}\, q)^{-1}\}+u_j\{-2(q\mathrm{cth}\, q-1)\}+u_{j-1}\{-\mathrm{th}(q/2)-$$
$$-(1-(q(\mathrm{sh}\, q)^{-1}\} = 0 \qquad (18)$$

The solution of (18) can be written as $u_j = C(g_1)^j + C_2(g_2)^j$, where

$$g_{1,2} = (q\mathrm{cth}\, q - 1) \pm (\mathrm{th} q/2) - q)[\mathrm{th}(q/2) - (1 - q(\mathrm{sh}\, q)^{-1})]^{-1}$$

and where C_1 and C_2 are arbitrary constants. So we obtain

$$u_j = C_1[(q\mathrm{cth}\, q - 1 + \mathrm{th}(q/2) - q)(\mathrm{th}(q/2) - (1 - q(\mathrm{sh}\, q)^{-1}))^{-1}]^j + \\ + C_2[(q\mathrm{cth}\ q - 1 - (\mathrm{th}(q/2) - q))(\mathrm{th}(q/2) - (1 - q(\mathrm{sh}\, q)^{-1}))^{-1}]^j$$

In the limiting case when $q \to 0, u_j \to (-1)^j$ as $j \to \infty$.

Thus all the solutions of the difference scheme are bounded as $j \to \infty$.

A linear operator is associated to the equation which connects the second derivatives and the spline values at three consequtive points, as follows:

$$L[u(x_j), h] = u(x_{j+1}) - 2u(x_j) + u(x_{j-1}) - \\ - h^2 q^{-2}[(1 - q(\mathrm{sh}\, q)^{-1})(u''_{j+1} + u''_{j+1}) + 2u''_j(-1 + q\mathrm{cth}\, q)] = \\ = u''_j h^2(1 - 2(q\mathrm{cth}(q/2))^{-1}) + u_j^{(iv)} h^4(1/12 - q^{-2}(1 - q(\mathrm{sh}\, q)^{-1})) + \\ + u_j^{(vi)} h^6(1/360 - (12q^2)^{-1}(1 - q(\mathrm{sh}\, q)^{-1})) + \cdots \tag{19}$$

When $q \to 0,\ L = o(h^4\varepsilon^{-1})$, when $\varepsilon = 1$ it has the order $o(h^4)$.

It is obvious from the form of the truncation error that we cannot increase the order beyond four.

When we apply the difference scheme (4) to the test equation

$$\varepsilon u'' = ku; \qquad u(0) = 0; \qquad u(1) = 0 \tag{20}$$

we obatin the characteristic equation given by:

$$u_{j+1}(q(\mathrm{sh}\, q)^{-1}) + u_j(-2q\mathrm{cth}\, q) + u_{j-1}(q(\mathrm{sh}\, q)^{-1}) = 0 \tag{21}$$

where $q = (k\varepsilon^{-1})^{1/2}h$. Solving (21) we get

$$u_j = A(\exp(q))^j + B(\exp(-q))^j \tag{22}$$

The theoretical solution of (20) is written as

$$u(x_j) = C_1(\exp(q))^j + C_2(\exp(-q))^j \tag{23}$$

Equation (22) gives the theoretical solution (23) and complete accuracy is obtained.

We may summarize the results obtained:

(a) Operator (15) with a choice of parameter $q \to 0$ is fourth order, with respect to h, for ε fixed, and stable when applied to the test equation (17). The solution values at an intermediate point can be calculated using adaptive spline.

(b) Operator (19) obtained from relation (4) when $q \to 0$ is fourth order if ε is fixed, and when it is applied to the test equation it gives complete accuracy.

4. Generation of Exponential Difference Scheme for Singular Perturbation Problem. The relation (4) is suitable for constructing an exponential difference scheme to solve numerically two point boundary value problems lacking the first derivative term. Now we discuss the solution of the differential equation

$$-\varepsilon u^{''} + p(x)u = f(x), \qquad p(x) > 0 \quad 0 < \varepsilon \ll 1, \tag{24}$$

$x \in [0,1]$, $p(x)$ and $f(x)$ being smooth enough.
The equation is subject to the boundary conditions

$$\begin{aligned} \alpha_1 u(0) + \beta_1 u'(0) = \gamma_1 \qquad \alpha_1\beta_1 \le 0 \\ \alpha_2 u(1) + \beta_2 u'(1) = \gamma_2 \qquad \alpha_2\beta_2 \ge 0 \end{aligned} \tag{25}$$

The solution of this problem displays the boundary layers at the end points $x = 0$ and $x = 1$ for "small" ε. In the boundary layers the solution of (24)–(25) has an exponential form. The exponential functions should be better suited than polynomials to follow the rapid variations that are typically found in the singular perturbation problems. So we have constructed an exponential basis for the adaptive splines and used it to obtain the collocation equation.
If we regard M_j as he spline second derivatives, the equation (24) becomes

$$-\varepsilon M_j + p_j u_j = f_j, \qquad p_j = p(x_j), \quad f_j = f(x_j).$$

By substituting M_j from the last equation into the scheme (4) we obtain:

$$\begin{aligned} &u_{j+1}\Big[1 - \frac{h^2}{q^2}\frac{p_{j+1}}{\varepsilon}\Big(1 - \frac{q}{\operatorname{sh} q}\Big)\Big] + u_j\Big[-2 - 2\frac{h^2}{q^2}\frac{p_j}{\varepsilon}(-1 + q\operatorname{cth} q\Big] + \\ &+u_{j-1}\Big[1 - \frac{h^2}{q^2}\frac{p_{j-1}}{\varepsilon}\Big(1 - \frac{q}{\operatorname{sh} q}\Big)\Big] = \Big[-\frac{f_{j+1}}{\varepsilon}\Big(1 - \frac{q}{\operatorname{sh} q}\Big) - 2\frac{f_j}{\varepsilon}(-1 + q\operatorname{cth} q) - \\ &-\frac{f_{j-1}}{\varepsilon}\Big(1 - \frac{q}{\operatorname{sh} q}\Big)\Big]\frac{h^2}{q^2} \end{aligned} \tag{26}$$

From (25) we obtain the first and last equation:

$$\begin{aligned} &u_0\Big\{\alpha_1 + \beta_1\Big[\frac{p_0}{\varepsilon}\Big(-\frac{h}{q}\operatorname{cth} q + \frac{h}{q^2}\Big) - \frac{1}{h}\Big]\Big\} + u_1\Big\{\beta_1\Big[\frac{p_1}{\varepsilon}\Big(\frac{h}{q\operatorname{sh} q} - \frac{h}{q^2}\Big) + \frac{1}{h}\Big]\Big\} = \\ &\qquad = \beta_1\Big[\frac{f_1}{\varepsilon}\Big(\frac{h}{q\operatorname{sh} q} - \frac{h}{q^2}\Big) + \frac{f_0}{\varepsilon}\Big(-\frac{h}{q}\operatorname{cth} q + \frac{h}{q^2}\Big)\Big] + \gamma_1 \\ &u_{n-1}\Big\{\beta_1\Big[\frac{p_{n-1}}{\varepsilon}\Big(-\frac{h}{q\operatorname{cth} q} + \frac{h}{q^2}\Big) - \frac{1}{h}\Big]\Big\} + u_n\Big\{\alpha_2 + \beta_2\Big[\frac{p_n}{\varepsilon}\Big(\frac{h}{q}\operatorname{cth} q - \frac{h}{q^2}\Big) + \frac{1}{h}\Big]\Big\} = \\ &\qquad = \gamma_2 + \beta_2\Big[\frac{f_n}{\varepsilon}\Big(\frac{h}{q}\operatorname{cth} q - \frac{h}{q^2}\Big) + \frac{f_{n-1}}{\varepsilon}\Big(-\frac{h}{q\operatorname{sh} q} + \frac{h}{q^2}\Big)\Big] \end{aligned} \tag{27}$$

5. Numerical Experiments and Conclusions. The algorythm for solving two point boundary value problems in this section was written in Fortran IV plus and executed on the Delta 340 (PDP-11/34). A double precission mode with 16 significant figures has been used.

In Table 1 is given the between real and approximate solution of he problem

$$-\varepsilon u'' + u = -\cos^2 \pi x - 2\varepsilon\pi^2 \cos 2\pi x$$

with boundary conditions $u(0) = u(1) = 0$, taken from [**2**].

This problem has the exact solution:

$$u(x) = (\exp(-(1-x)/\sqrt{\varepsilon}) + \exp(-x/\sqrt{\varepsilon}))/(1 + \exp(-1/\sqrt{\varepsilon})) - \cos^2 \pi x.$$

The scheme (26–27) is applied to this problem on the equidistant grid with N subintervals in the interval $[0, 1]$, for different $\varepsilon.(q = 1\sqrt{\varepsilon})$.

Table 1

$\varepsilon \backslash N$	32	64	128	256	512	1024
1/64	0.159E–02	0.401E–03	0.100E–03	0.251E–04	0.627E–05	0.157E–05
1/1000	0.775E–02	0.297E–03	0.915E–03	0.254E–04	0.624E–05	0.157E–05

These results compare favourably with results obtained by using cubic spline exponentially fitted (see [**5**]). We can conclude that adaptive splines give better results. Exponential basis of adaptive spline is more suited to the problem (24–25) than exponential spline which is obtained when polynomial spline is applied to the differential equation exponentially fitted.

In Table 2 it is computationally shown thas the scheme (26–27) achieves a second order convergence of uniform accuracy over the uniform mesh.

Table 2

$\varepsilon \backslash k$	1	2	3	4	5	Py
2^0	2.00	2.00	2.00	2.00	2.00	2.00
2^{-1}	2.00	2.00	2.00	2.00	2.00	2.00
2^{-2}	2.00	2.00	2.00	2.00	2.00	2.00
2^{-3}	1.99	1.99	2.00	2.00	2.00	2.00
2^{-4}	1.96	1.99	2.00	2.00	2.00	1.99
2^{-5}	1.95	1.98	2.00	2.00	2.00	1.99
2^{-6}	1.92	1.97	1.99	2.00	2.00	1.98
2^{-7}	1.85	1.96	2.00	2.00	2.00	1.96
2^{-8}	1.73	1.93	1.98	1.99	2.00	1.93
2^{-9}	1.52	1.86	1.96	1.99	2.00	1.87

Here is given the test of uniform convergence when the scheme (26–27) is applied to the problem

$$-\varepsilon u^{''} + (1+x)^2 u = (4x^2 - 14x + 4)(1+x)^2$$

with boundary conditions $u(0) - u'(0) = 0,\ u(1) = 0$.

This example and notation is taken from [**2**]. The errors z_s and the rates p_i of uniform convergence is based on double mesh principle (see [**2**]).

If N is a number of subintervals of interval $[0,1]$ we define

$$z_{s0} = \max_{0\le i\le n} \mid u_i^N - u_{2i}^{2N} \mid \ \text{ and } \ p_i = (\ln\, z_{s-1} - \ln\, z_s)/\ln\, 2;$$

z_s and z_{s0} correspond to maximum error between two consequtive meshes (for $h = 1/N$ and $h = 1/2N$ respectively).

For parameter q in (26–27) we used the variable parameter $q_i = (p_i/\varepsilon)^{1/2} h$, where $p_i = p(x_i)$.

Table 3 contains the maximmum error z_s for the example

$$-\varepsilon u^{''} + (2x^3 - 3x^2 + 6)u = 4(3x^2 - 3x + 2)((x-0.5)^2 + 2),$$

taken from [**2**] too, with boundary conditions $u(0) = -1,\ \ u(1) = 0$.

Table 3

$\varepsilon \backslash k$	1	2	3	4	5
2^0	0.310E–03	0.774E–03	0.194E–04	0.484E–05	0.121E–05
2^{-1}	0.452E–03	0.113E–03	0.283E–04	0.706E–05	0.177E–05
2^{-9}	0.861E–03	0.224E–03	0.566E–04	0.142E–04	0.355E–05
2^{-10}	0.851E–03	0.231E–03	0.592E–04	0.149E–04	0.373E–05
10^{-2}	0.853E–03	0.231E–03	0.591E–04	0.149E–04	0.372E–05
10^{-5}	0.175E–03	0.901E–04	0.415E–04	0.146E–04	0.413E–05

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