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UNIFORM c-CONVEXITY OF $\mathbf{L}^p$, 0<p≤1

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Abstract. We extend a result of Globevnik by proving that L^p spaces with $0 < p \leq 1$ are uniformly c-convex. We also give the precise values for the moduli of c-convexity of L^p. A short proof of Globenik's result is included.

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1. Introduction. A result of Thorp and Whitley [**8**] states that L^1-spaces are strictly c-convex, although the unit sphere of $L^1(0,1)$ does not possess exstreme points. This results was strenghtened by Globevnik [**1**], who proved that L^1-spaces are uniformly c-convex. Further examples of uniformly c-convex normed spaces are given in [**6**]. However, it seems that the case of quasi-normed spaces has not been discussed yet. In this paper we present some results in this direction. Theorems 1, 2, 3 were proved by the author in [**5**].

Definition. A complex quasi-normed space X, i. e. a complex linear space with a quasi-norm $\|\cdot\|$, is said to be uniformly c-convex if there exists a real function δ on $[0,+\infty)$ such that $\delta(\varepsilon) > 0$ whenever $\varepsilon > 0$, and

$$\delta(\varepsilon) \leq \sup\{\|x+\lambda y\| : \ |\lambda| \leq 1\} - 1 \tag{1}$$

for all x, y with $\|x\| = 1, \ \|y\| \geq \varepsilon$. The supremum of all δ, satisfying (1), is denoted by δ_X^c and is called the modulus of c-convexity of X.

We recall that a quasi-norm $\|\cdot\|$ on a linear space X has the following properties: 1. $\|x\| \geq 0$, 2. $x = 0$ if $\|x\| = 0$, 3. $\|\lambda x\| = |\lambda| \, \|x\|$ for all scalars λ, 4. there exists a $K \geq 1$ such that $\|x+y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$. If the quasi-norm is p-subadditive for some p, $0 < p \leq 1$, i. e. if $\|x+y\|^p \leq \|x\|^p + \|y\|^p$, then X is called a p-normed space.

We consider the complex Lebesgue space $L^p = L^p(m)$, $0 < p \leq 1$, where m is a positive measure on a σ-algebra of subset of a set S. The quasi-norm on L^p is given by

$$\|x\| = \|x\|_p := Bigl\{\int_S |x|^p \, dm\Bigr\}^{1/q}$$

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The modulus of c-convexity of L^p will be denoted by δ_p. Our main results is the following theorem.

THEOREM 1. *The space $L^p, 0 < p \leq 1$ is uniformly c-convex. Moreover,*

$$\delta_p(\varepsilon) \geq F_p(\varepsilon) := -1 + \left\{ \frac{1}{2\pi} \int_0^{2\pi} \mid 1 + \varepsilon e^{it} \mid^p dt \right\}^{1/p}, \quad \varepsilon \geq 0 \tag{2}$$

with equality if L^p is infinite-dimensional.

The inequality (2) is a consequence of the following stronger result.

THEOREM 2 *If $x, y \in L^p$, $0 < p \leq 1$, then*

$$\int_0^{2\pi} \|x + e^{it} y\|^p dt \geq \int_0^{2\pi} \left| \|x\| + e^{it} \|y\| \right|^p dt$$

Note that the same inequality is valid for $p \in [1, 2]$. A proof can be found in [**7**], but the arguments given there cannot be applied in the case $0 < p < 1$. On the other hand, the proof of Theorem 2, which will be given in Section 2, works for all $p \in (0, 2]$. It is a natural question whether the modulus δ_n can be improved by use an equivalent quasi-norm. The following theorem gives a partial answer to this queston.

THEOREM 3. *Let the space L^p, $0 < p \leq 1$, be infinite-dimensional. If a p-normed space X is isomorphic to L^p, then $\delta_X^c(\varepsilon) \leq F_p(\varepsilon)$ for every $\varepsilon \geq 0$.*

As an immediate consequence of Theorem 3 and the inequality $F_p(\varepsilon) < F_q(\varepsilon)$, $p < q$, $\varepsilon > 0$, we have the following well known fact.

COROLLARY. *If an infinite-dimensional L^q space is isomorphic to L^p, $0 < p$, $q < 1$, then $p = q$.*

In Section 3 we give some more applications of Theorem 3.

2. Proofs of the theorems. The proof of Theorem 2 is based on the following lemma.

LEMMA 1. *Let $0 < p \leq 1$. Then the function φ, given by*

$$\varphi(u, v) = \int_0^{2\pi} \mid u^{1/p} + v^{1/p} e^{it} \mid^p dt,$$

is convex on the set $\{(u, v) : u \geq 0, \ v \geq 0\}$.

Proof. Since φ is continuous and $\varphi(cu, cv) = c\varphi(u, v)$ for all $c > 0$, it is enough to prove that the function $\psi(\varepsilon) : = \varphi(1, \varepsilon)$ is convex on the interval $[0, \infty)$. Suppose first that $0 \leq \varepsilon \leq 1$. Then

$$\psi(\varepsilon) = \int_0^{2\pi} \mid (1 + \varepsilon^{1/p} e^{it})^{p/2} \mid^2 dt$$

Hence, by Parseval's formula applied to the function $t \mapsto (1+\varepsilon^{1/p}e^{it})^{p/2} = \sum \binom{p/2}{n} \varepsilon^{n/p} e^{int}$,

$$\psi(\varepsilon) = 2\pi\left(1 + \sum_{n=1}^{\infty} \binom{p/2}{n}^2 \varepsilon^{2n/p}\right).$$

From this it follows that ψ is convex on $[0,1]$ as a sum of convex functions. Now we can prove that ψ is convex on $(1,+\infty)$. Indeed, if $\varepsilon > 1$, we use the equality $\psi(\varepsilon) = \varepsilon\psi(1/\varepsilon)$ to obtain $\psi''(\varepsilon) = \varepsilon^{-3}\psi''(1/\varepsilon) > 0$. Finally, it is enough to prove that $\psi(\varepsilon)$ is differentiable for $\varepsilon = 1$.

Let

$$f(\varepsilon) = \psi(\varepsilon^p) = \int_0^{2\pi} (1+\varepsilon^2+2\varepsilon\cos\, t)^{p/2} dt, \quad \varepsilon > 0.$$

By Leibniz's rule,

$$f'(\varepsilon) = p\int_0^{2\pi} (\varepsilon + \cos\, t)(1+\varepsilon^2+2\varepsilon\cos\, t)^{p/2-1} dt$$

if $\varepsilon \neq 1$. Since $(\varepsilon+\cos\, t)^2 \leq (\varepsilon+\cos\, t)^2 + \sin^2 t = 1+\varepsilon^2+2\varepsilon\cos\, t$ we have

$$\begin{aligned} |\, \varepsilon + \cos\, t \,|\, (1+\varepsilon^2+2\varepsilon\cos\, t)^{p/2-1} &\leq [(\varepsilon+\cos\, t)^2 + \sin^2 t]^{(p-1)/2} \leq (\sin^2 t)^{(p-1)/2} \\ &= |\sin\, t\,|^{p-1}. \end{aligned}$$

Hence, by the Lebesgue dominated couvergence theorem, $\lim\limits_{\varepsilon\to 1} f(\varepsilon)$ exist and is finite. This completes the proof.

Proof of Theorem 2. Let $x, y \in L^p$, $0 < p \leq 1$. Then the support of $|\,x\,| + |\,y\,|$ is of σ-finite measure. So we can apply Fubini's theorem to get

$$\int_0^{2\pi} \|x + e^{it}y\|^p dt = \int_S dm \int_0^{2\pi} |\, x + e^{it}y \,|^p\, dt = \int_S \varphi[|\,x\,|^p, |\,y\,|^p] dm,$$

where we have used the equality

$$\int_0^{2\pi} |\, x + e^{it}y \,|^p\, dt = \int_0^{2\pi} \Big|\, |\,x\,| + e^{it}\,|\,y\,| \,\Big|^p dt.$$

Hence, by Jensen's inaquality and Lemma 1,

$$\begin{aligned} \int_0^{2\pi} \|x + e^{it}y\|^p dt \geq& \varphi\left[\int_S |\,x\,|^p\, dm, \int_S |\,y\,|^p\, dm\right] \\ =& \varphi[\|x\|^p, \|y\|^p] = \int_0^{2\pi} \Big|\, \|x\| + e^{it}\|y\| \,\Big|^p dt. \end{aligned}$$

Remark. In the case of L^1 a short proof of Theorem 2 can be given in the following way. Let $x, y \in L^1$. Then

$$\begin{aligned} \int_0^{2\pi} \|x + e^{it}\| dt &= \int_0^{2\pi} \Big\|\, |\,x\,| + e^{it}\,|\,y\,| \,\Big\| dt \geq \\ &= \int_0^{2\pi} \Big|\, \int_S (|\,x\,| + e^{it}\,|\,y\,|) dm \,\Big| dt = \int_0^{2\pi} \Big|\, \|x\| + e^{it}\|y\| \,\Big| dt. \end{aligned}$$

Proof of Theorem 1. The inequality (2) follows easily from Theorem 2. To prove the rest suppose that L^p is infinite-dimensional. Then, by Proposition I. 5 of [**3**], L^p contains an isometric copy of the sequence space. Thus the assertion reduces to the case l^p.

Let $\{e_k\}_0^\infty$ be the standard basis of l^p. For a positive integer n let $m = 2^n,\ \varepsilon > 0$ and

$$x = m^{-1/p} \sum_{k=0}^{m=1} e_k, \quad y = \varepsilon m^{-1/p} \sum_{k=0}^{m=1} e^{2k\pi i/m} e_k.$$

Since $\|x\| = 1,\ \|y\| = \varepsilon$, we have

$$[1 + \delta_p(\varepsilon)]^p \leq \max_{|\lambda|=1} \|x + \lambda y\|^p,$$

where we have used the fact that the function $\lambda \mapsto \|x + \lambda y\|^p$ is supharmonic. On the other hand, one can choose $t_m \in [0, 2\pi/m]$ so that

$$\max_{|\lambda|=1} \|x + \lambda y\|^p = m^{-1} \sum_{k=0}^{m-1} \mid 1 + \varepsilon e^{it_m} e^{2k\pi i/m} \mid^p .$$

Hence

$$[1 + \delta_p(\varepsilon)]^p \leq m^{-1} \sum_{k=0}^{m-1} \mid 1 + \varepsilon e^{itm,k} \mid^p,$$

where $2k\pi/m \leq t_{m,k} \leq 2(k+1)\pi/m$. Now the resuly follows from the fact that the last sum tends to

$$(2\pi)^{-1} \int_0^{2\pi} \mid 1 + \varepsilon e^{it} \mid^p dt.$$

For the proof of Theorem 3 we need the following propositio. It is an extension of the corresponding result for the space l^1 [**4**, Proposition 2. e. 3].

PROPOSITION 1. *Let X be a p-normed space which is isomorphic to l^p, $0 < p \leq 1$. Then, for every $c > 1$, there exists a linear operator $T : l_p \to X$ such that $c^{-1}\|x\| \leq \|Tx\| \leq c\|x\|$ for all $x \in l^p$.*

Proof. The proof is the same as that of Proposition 3 e. 3 of [**4**]. Let S be an isomorphism of l^p onto X and assume, without loss of generality that $\alpha\|Sx\| \leq \|x\| \leq \|Sx\|$, for some $\alpha > 0$ and all $x \in l^p$. Let $c > 1$ and let $\{P_n\}_{n=1}^\infty$ be the projections induced by the unit vector basis $\{e_n\}$ of l^p:

$$P_n x = \sum_{j=1}^{n} a_j e_j, \quad x = \sum_{n=1}^{\infty} a_n e_n \in l^p.$$

For every n put $\lambda = \sup\{\|x\| : \|Sx\| = 1,\ P_n x = 0\}$. Then $\lambda_n \downarrow \lambda$ for some $\lambda,\ \alpha \leq \lambda \leq 1$. Let N be such that $\lambda_N < \lambda\sqrt{c}$. By the definition of $\{\lambda_n\}$ there are vectors $\{y_k\}_{k=1}^{\infty}$ such that, for all k, $\|Sy_k\| = 1$, $P_N y_k = 0$, $\|y_k\| > \lambda/\sqrt{c}$ and $\mathrm{supp}(y_m) \cap \mathrm{supp}(y_k) = \varnothing$ for $m \neq k$. For every choice of scalars $\{a_k\}_{k=1}^{\infty}$ we have

$$P_N\left(\sum_{k=1}^{\infty} a_k y_k\right) = 0$$

and hence, by the definition of λ_N,

$$\begin{aligned}\left\| S\sum_{k=1}^{\infty} a_k y_k \right\| &\ge \lambda_N^{-1} \left\| \sum_{k=1}^{\infty} a_k y_k \right\| = \lambda_N^{-1}\left(\sum_{k=1}^{\infty} |\, a_k\,|^p \, \|y_k\|^p\right)^{1/p} \\ &\ge \lambda_N^{-1} c^{-1/2} \lambda \left(\sum_{k=1}^{\infty} |\, a_k\,|^p\right)^{1/p} \ge c^{-1}\left(\sum_{k=1}^{\infty} |\, a_k\,|^p\right)^{1/p}.\end{aligned}$$

On the other hand, since X is a p-normed space, we have

$$\left\| S\sum_{k=1}^{\infty} a_k y_k \right\|^p \le \sum_{k=1}^{\infty} |\, a_k\,|^p \, \|Sy_k\|^p = \sum_{k=1}^{\infty} |\, a_k\,|^p \; .$$

The desired operator is defined by $Te_k = Sy_k$, $k = 1, 2, \ldots$.

Proof of Theorem 3. Let $c > 1$ and let X be an infinite-dimensional p-normed space isomorphic to L^p. Since X contains an isomorphic copy of l^p, there is a linear operator $T : l^p \to X$ such that $c^{-1}\|x\| \le \|Tx\| \le c\|x\|$ for all $x \in l^p$. For a fixed $\varepsilon > 0$ there are $x, y \in l^p$ such that $\|x\| = 1$, $\|y\| \ge c^2\varepsilon$ and

$$\sup_{|\lambda| \le 1} \|x + \lambda y\| \le c[1 + F_p(c^2\varepsilon)].$$

Let $x' = Tx/\|Tx\|$, $y' = Ty/\|Tx\|$. Then $\|x'\| = 1$ and $\|y'\| \ge \varepsilon$, because $\|Tx\| \le c$, $\|Ty\| \ge c^{-1}\|y\| \ge c\varepsilon$. Hence, by the definition of δ_X^c,

$$1 + \delta_X^c(\varepsilon) \le \sup_{|\lambda| \le 1} \|x' + \lambda y'\|$$

On the other hand, $\|x' + \lambda y'\| \le c^2\|x + \lambda y\| \le c^3[1 + F_p(c^2\varepsilon)]$. This implies

$$1 + \delta_X^c(\varepsilon) \le c^3[1 + F_p(c^2\varepsilon)].$$

Since $c > 1$ was arbitrary, we get $\delta_X^c(\varepsilon) \le F_p(\varepsilon)$.

3. Uniform c-convexity in $\mathbf{l}^p$. In this section we given an extension of Theorem 1 to subspaces of l^p.

THEOREM 4. *Let X be an infinite-dimensional subspaces of l^p, $0 < p \le 1$. Then $\delta_X^c(\varepsilon) = \delta_{l^p}^c(\varepsilon$ for all $\varepsilon > o$.*

In the case $p = 1$ this result follows directly from Theorem 3 and the fact that for every closed infinite-dimensional subspace X of l^p, $1 \le p < \infty$, there is an isomorpism of l^p into X [**4** Propositional 2. a. 2]. To prove Theorem 4 for $p < 1$ we use a similar but somewhat more general approach .

PROPOSITION 2. *Let X be a closed infinite-dimensional subspace of l^p, $0 < p < \infty$. Then, for every $c > 1$, there is a linear operator $T : l^p \to X$ such that $c^{-1}\|x\| \le \|Tx\| \le c\|x\|$ for all $x \in l^p$.*

Proof.. We proceed in the same way as in [**4**, Propositions 1. a. 11 and 1. a.9]. Let $c > I$. For any $b > 0$ we find two sequences, $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$, such that: 1. $x_n \in X$, 2.$\|x_n\| = \|y_n\| = 1$, 3.$\|x_n - y_n\| \le b/2^n$, and 4. supp$(y_m) \cap$ supp $(y_n) = \varnothing$ for $m \ne n$. From the last condition it follows that Y: $= [y_n]_{n=0}^{\infty}$, the closed linear span of $\{y_n\}$, isometrically isomorphic to l^p. Thus it is enough to find an operator S:$Y \to X$ such that $c^{-1}\|y\| \le \|Sy\| \le c\|y\|$, $y \in Y$.

Let $q = \min(p, 1)$ and choose b so that $b^q(1 - 1/2^q)^{-1} = 1 - 1/c^q$. For $y = \sum_{n=0}^{\infty} a_n y_n$ let $Sy = \sum_{n=0}^{\infty} a_n x_n$ and $Uy = y - Sy$. Then

$$\|Uy\|^q \le \sum_{n=0}^{\infty} \mid a_n \mid^q \|x_n - y_n\|^q \le \|y\|^q \sum_{n=0}^{\infty} \|x_n - y_n\|^q \le b^q(1 - 1/2^q)^{-1}\|y\|^q,$$

where we used the condition 3. Hence

$$\|Sy\|^q = \|y - Uy\|^q \le \|y\|^q + \|Uy\|^q \le c^q\|y\|^q.$$

On the other hand, since $y = \sum_{n=0}^{\infty} U^n Sy$, we have

$$\|y\|^q \le \|Sy\|^q \sum_{n=0}^{\infty} \|U\|^{nq} \le c^q\|Sy\|^q.$$

This completes the proof.

Using Proposition 2 we can prove that Theorem 4 holds for every $p > 0$. If X is closed, this can be done in the same way as in the proof of Theorem 3. If X is not closed, one can not closed, one can use the equality $\delta_X^c = \delta_Y^c$, where Y is the closure of X. We note that, if $p > 2$, the modulus of c-convexity of l^p is equal to $(1 + \varepsilon^p)^{1/p} - 1$. This follows from Clarkson's inequality [**2**]:

$$\|x + y\|^p + \|x - y\|^p \ge 2(\|x\|^p + \|y\|^p),\ x, y \in L^p,\ p > 2.$$

4. Remarks. One of simple ways to prove that $L^p(m)$ is uniformly c-convex is to use the inequality

$$(2\pi)^{-1} \int_0^{2\pi} \mid u + ve^{it} \mid^p dt \ge (\mid u \mid^2 + p \mid v \mid^2 /2)^{p/2}, \quad 0 < p < 2, \tag{3}$$

valid for all complex numbers u, v. Indeed, if $0 < p < 2$, the function $N(u, v) := (\mid u \mid^{2/p} + p \mid v \mid^{2/p} /2)^{p/2}$ is a norm and, consequently,

$$\int_S N(\mid x \mid^p, \mid y \mid^p) dm \ge N\left(\int_S \mid x \mid^p dm, \int_S \mid y \mid^p dm\right) = (\|x\|^2 + p\|y\|^2/2)^{p/2},$$

where $x, y \in L^p(m)$. Hence, by (3),

$$(2\pi)^{-1} \int_0^{2\pi} \|x + ye^{it}\|^p dt \geq (\|x\|^2 + p\|y\|^2/2)^{p/2}.$$

This gives the estimate $\delta_p(\varepsilon) \geq (1 + p\varepsilon^2/2)^{1/2} - 1$.

To prove the inequality (3) we may assume that $u = 1$. Then, if $| v | \leq 1$, by Parseval's formula,

$$f(v) := (2\pi)^{-1} \int_0^{2\pi} | 1 + ve^{it} |^p \, dt \geq 1 + p^2 | v |^2 /4 \geq (1 + p | v |^2 /2)^{p/2}.$$

If $| v | > 1$, we have

$$f(v) = | v |^p \, f(1/v) \geq | v |^p \, (1 + p/(2 | v |^2))^{p/2} \geq (1 + p | v |^2 /2)^{p/2}.$$

After completing this paper the author has learned of a recent paper of Davis, Garling and Tomczak-Jaegermann [**9**]. For a quasi-normed space X (with some additional propeties) they define the moduli H_q^X, $0 < q \leq \infty$, and $I_{q,r}(X)$, $0 < q \leq \infty$, $2 \leq r < \infty$, in the following way:

$$1 + H_q^X(\varepsilon) = \inf \left\{ \left(\frac{1}{2\pi} \int_0^{2\pi} \|x + e^{it}y\|^q dt \right)^{1/q} : \|x\| = 1, \ \|y\| = \varepsilon \right\}, \ \ \varepsilon \geq 0;$$

$I_{q,r}(X)$ is the largest non-negative λ such that

$$\left(\frac{1}{2\pi} \int_0^{2\pi} \|x + e^{it}y\|^q dt \right)^{1/q} \geq (\|x\|^r + \lambda\|y\|^r)^{1/r}$$

for all $x, y \in X$.

In [**9**] the following problem is raised (Problem 4): Is it true that $I_{q,2}(C) = q/2$ for $q < 2$, where C is the complex plane? The preceding remarks show that the answer is yes. Moreover, we have the following results.

Theorem 5. *Let X be an infinite-dimensional L^p-space or an infinite-dimensional subspace of l^p, $0 < p \leq 2$. Then:* 1. *$H_q^X(\varepsilon) = F_p(\varepsilon)$ if $q \geq p$, and* 2. *$H_q^X(\varepsilon) = F_q(\varepsilon)$ if $0 < q \leq p$.*

The first equality follows from Theorems 1, 2 and 4 because H_q^X increases with q and $H_\infty^X = \delta_X^c$. To prove the second equality one can use the inequality

$$\int_0^{2\pi} \|x + e^{it}y\|_p^q dt \geq \int_0^{2\pi} \Big| \, \|x\|_p + e^{it} \, \|y\|_p \Big|^q dt$$

($q \leq p \leq 2$), which follows from Theorem 2 and the fact that every finite-dimensional L^p-space is isometric to a subspace of $L^q(\mu)$, for some measure μ [**10**, Lemma 21. 1. 3.].

Note that if $q \leq 2$ then Theorem 5 holds for every (non-trivial) L^p-space.

Theorem 6. *Under the hypothesis of Theorem* 5 *we have* $I_{q,2}(X) = p/2$ *for* $q \geq p$, *and* $I_{q,2}(X) = q/2$ *for* $q \leq p$.

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