

ON A PROBLEM OF ERDÖS AND IVIĆ

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Abstract. Let us usual $\omega(n)$ and $\Omega(n)$ denote the number of distinct prime factors and the number of total prime factors of n respectively. Asymptotic formulas for the sum $\sum_{2 \leq n \leq x} n^{-1/\Omega(n)}$ and the logarithm of the sum $\sum_{2 \leq n \leq x} n^{-1/\omega(n)}$ are derived.

1. Introduction and statement of results. Let as usual $\omega(n)$ and $\Omega(n)$ denote the number of distinct prime factors and the number of total prime factors of n . In [1] Erdős and Ivić proved that for $x \geq x_0$ and some constants $0 < c_2 < c_1$, the following inequality holds

$$x \exp\{-c_1(\log x \log_2 x)^{1/2}\} \leq \sum_{2 \leq n \leq x} n^{-1/\omega(n)} \leq x \exp\{-c_2(\log x \log_2 x)^{1/2}\},$$

where $\log_2 x = \log \log x$. In [1] they also proved that for $x \geq x_0$ and some constants $0 < c_4 < c_3$ the inequality

$$x \exp\{-c_3(\log x)^{1/2}\} \leq \sum_{2 \leq n \leq x} n^{-1/\Omega(n)} \leq x \exp\{-c_4(\log x)^{1/2}\}$$

holds. Then they conjecture that for some $c_5, c_6 > 0$, one has

$$(1.1) \quad \sum_{2 \leq n \leq x} n^{-1/\omega(n)} = x \exp\{-(c_5 + o(1))(\log x \log_2 x)^{1/2}\},$$

and

$$(1.2) \quad \sum_{2 \leq n \leq x} n^{-1/\Omega(n)} = x \exp\{-(c_6 + o(1))(\log x)^{1/2}\}.$$

The aim of this note is to give an asymptotic formula for sum in (1.2) and an asymptotic formula for the logarithm of sum in (1.1), respectively. Our results may be stated as follows.

THEOREM 1.

$$\begin{aligned} \sum_{2 \leq n \leq x} n^{-1/\Omega(n)} &= \\ &= C \pi^{1/2} (\log 2)^{-3/4} \cdot x (\log x)^{5/4} \cdot \exp\{-2(\log 2 \log x)^{1/2}\} (1 + O((\log x)^{-1/4})), \end{aligned}$$

where

$$(1.3) \quad C = (1/4) \prod_{p>2} (1 + 1/(p(p-2))) = 0.378694.$$

THEOREM 2.

$$\begin{aligned} \sum_{2 \leq n \leq x} n^{-1/\omega(n)} &= \\ &= x \exp\left\{-2(\log x \log_2 x)^{1/2} \cdot \left(1 - \frac{3 \log_3 x}{2 \log_2 x} + \frac{3 \log 2 - 2}{2 \log_2 x} - \frac{\log_3 x}{\log_2^2 x} + O\left(\frac{1}{\log_2^2 x}\right)\right)\right\}. \end{aligned}$$

Thus the above two theorems settle affirmatively the conjecture of Erdős and Ivić in question.

Notations. Let x be a positive variable assumed to be sufficiently large,

$$N(x, k) = |\{n \leq x \mid \Omega(n) = k\}|, \quad \pi(x, k) = |\{n \leq x \mid \omega(n) = k\}|$$

$$A = x \exp\{-10(\log x)^{1/2}\}, \quad B = x \exp\{-10(\log x \log_2 x)^{1/2}\},$$

$$P = (\log x)^{1/2}, \quad Q = (\log x)^{1/2} (\log_2 x)^{-1/2}, \quad L = (\log x \log_2 x)^{1/2},$$

$$M = (\log 2 \log x)^{1/2}, \quad N = (\log 2)^{-1/2} (\log x)^{1/2}.$$

2. Proof of Theorem 1. To prove this we need

LEMMA 1 [2]. *For $x \geq 3$ and $5 \log_2 x \leq k \leq \log x / \log 2$ we have uniformly*

$$N(x, k) = C(x/2^k) \log(x/2^k) + O((x/2^k) \log_2(3x/2^k)),$$

where C is defined by (1.3).

Proof of Theorem 1. We shall use an inequality of Erdős and Ivić [1, p. 52]:

$$\sum_{n \leq x, \Omega(n) \geq k} 1 \ll x \cdot 2^{-k/4}, \quad \text{if } k \geq (\log_2 x)^2,$$

where the \ll -constant is uniform in k and x . From this we have

$$\sum_{n \leq x, \Omega(n) \geq 40P} n^{-1/\Omega(n)} \ll x e^{-4P}$$

It is evident that

$$\sum_{A < n \leq x, \Omega(n) \leq 0.1P} n^{-1/\Omega(n)} \ll x e^{-4P}.$$

From the above two estimates we get

$$(2.1) \quad W(x) := \sum_{2 \leq n \leq x} n^{-1/\Omega(n)} = \sum_{\substack{A < n \leq x \\ 0.1P < \Omega(n) \leq 40P}} n^{-1/\Omega(n)} + O(xe^{-4P}) = W' + O(xe^{-4P}), \text{ say.}$$

Lemma 1 implies that

$$N(n, k) = C(n/2^k) \log(n/2^k) + r_n, \text{ where } r_n = O(n/2^k \log_2(3n/2^k)).$$

Therefore we have

$$(2.2) \quad \begin{aligned} W' &= \sum_{0.1P < k \leq 40P} \sum_{A < n \leq x} n^{-1/k} (N(n, k) - N(n-1, k)) \\ &= C \sum_{N < k \leq 40P} 2^{-k} \sum_{A < n \leq x} n^{-1/k} \log n + C \sum_{0.1P < k \leq N} 2^{-k} \sum_{A < n \leq x} n^{-1/k} \log n \\ &+ \sum_{0.1P < k \leq 40P} \sum_{A < n \leq x} n^{-1/k} (r_n - r_{n-1}) + O\left(\sum_{0.1P < k \leq 40P} k \cdot 2^{-k} \sum_{A < n \leq x} n^{-1/k} \right) \\ &= W_1 + W_2 + W_3 + O(W_4), \text{ say.} \end{aligned}$$

First we estimate W_1 in (2.2). We shall use the following inequality [3, Ch. 5, Theorem 8.2]

$$(2.3) \quad \left| \sum_{a \leq n \leq \xi} f(n) - \int_a^\xi f(v) dv - \alpha \right| \leq f(\xi - 1), \text{ if } \xi \geq a + 1,$$

where $f(x)$ is decreasing, $f(x) \rightarrow 0$, if $x \rightarrow \infty$ and $0 \leq \alpha \leq f(a)$. Alternatively, we may use the Euler-Maclaurin summation formula.

By this we have

$$\begin{aligned} \sum_{A < n \leq x} n^{-1/k} \log n &= \int_A^x z^{-1/k} \log z dz + O(A^{-1/k} \log A) \\ &= (x \log x) \cdot x^{-1/k} (1 + O(P^{-1})). \end{aligned}$$

So

$$(2.4) \quad W_1 = Cx \log x \sum_{N < k \leq 40P} 2^{-k} x^{-1/k} (1 + O(P^{-1})).$$

Using (2.3) again, we have

$$(2.5) \quad \sum := \sum_{N < k \leq 40P} 2^{-k} x^{-1/k} = \int_N^{40P} 2^{-z} x^{-1/z} dz + O(e^{-2M}).$$

Setting $z = Nu^2$, we have

$$\sum = N \int_1^\infty \exp\{-M(u^2 + u^{-2})\} \cdot 2udu + O(e^{-2M}).$$

Setting further $M^{1/2}(u - u^{-1}) = v$, we obtain

$$\sum = Ne^{-2M} \int_0^\infty e^{-v^2} \left(\frac{v}{M} + \frac{v^2 + 2M}{M(v^2 + 4M)^{1/2}} \right) dv + O(e^{-2M}).$$

Using Lagrange's mean value theorem, we have

$$|(v^2 + 4M)^{-1/2} - (1/2)M^{-1/2}| \leq (1/2)v^2M^{-3/2},$$

so

$$(2.6) \quad \begin{aligned} \sum &= NM^{-1/2}e^{-2M} \left(\int_0^\infty e^{-v^2} dv + O(M^{-1/2} \int_0^\infty e^{-v^2} \cdot v dv) \right) \\ &= (1/2)\pi^{1/2}NM^{-1/2}e^{-2M} (1 + O(P^{-1/2})). \end{aligned}$$

Combining (2.4) — (2.6) we get

$$(2.7) \quad \begin{aligned} W_1 &= (1/2)C\pi^{1/2}(\log 2)^{-3/4}x(\log x)^{5/4} \exp\{-2(\log 2 \cdot \log x)^{1/2}\} \cdot \\ &\quad (1 + O((\log x)^{-1/4})). \end{aligned}$$

In the same way we obtain

$$(2.8) \quad \begin{aligned} W_2 &= (1/2)C\pi^{1/2}(\log 2)^{-3/4}x(\log x)^{5/4} \exp\{-2(\log 2 \cdot \log x)^{1/2}\} \cdot \\ &\quad (1 + O((\log x)^{-1/4})). \end{aligned}$$

We turn now to estimate W_3 in (2.2). We have

$$\begin{aligned} W_3 &= \sum_{0.1P < k \leq 40P} \sum_{A < n \leq x} n^{-1/k} (r_n - r_{n-1}) \\ &= \sum_k \sum_n r_n (n^{-1/k} - (n+1)^{-1/k} + [x]^{-1/k} r_{[x]} - ([A] + 1)^{-1/k} r_{[A]}) \\ &\ll \sum_k \sum_n k^{-1} \cdot 2^{-k} n^{-1/k} \log_2 x + x \sum_k 2^{-k} x^{-1/k} \log_2 x \\ &\ll (\log x)^{-3/2} \log_2 x \sum_k 2^{-k} \sum_n n^{-1/k} \log n + x \log_2 x \sum_k 2^{-k} x^{-1/k} \\ &\ll (\log x)^{-3/2} \log_2 x \cdot (W_1 + W_2) + x \log_2 x \cdot (\sum). \end{aligned}$$

Combining (2.6)–(2.8) we get

$$(2.9) \quad W_3 \ll x(\log x)^{1/4} \log_2 x e^{-2M}.$$

It is easy to prove that $W_4 \ll x(\log x)^{3/4} e^{-2M}$. From this and (2.1), (2.2), (2.7), (2.8) and (2.9), the Theorem follows.

3. Proof of Theorem 2. To prove Theorem 2 we shall need the following result on the function $\pi(x, k)$.

LEMMA2. [4] Let $U = U(x, k) = \log_2 x - \log k - \log_2 k$. Then the estimate

$$\pi(x, k) = \frac{x}{k!} \exp \left\{ k \left(\log U + \frac{\log U}{U} + O\left(\frac{1}{U}\right) \right) \right\}$$

holds uniformly in the range $x \geq 3$, $(\log_2 x)^2 \leq k \leq \log x / 3 \log_2 x$.

Proof of Theorem 2. We shall use a classical inequality of Hardy and Ramanujan [5, p. 265]:

$$\sum_{n \leq k, \Omega(n)=k} 1 < Ex(\log^{-1} x)(\log_2 x + F)^k / k!.$$

where $E, F > 0$ are absolute constants, $x \geq 2$ and $k \geq 0$.

From this we have

$$\begin{aligned} \sum_{n \leq x, \Omega(n) \geq k} 1 &\ll x \sum_{r \geq k} \exp\{r \log_3 x - r \log r + O(\gamma)\} \\ &\ll x \exp\{-k \log k + O(k \log_3 x)\}. \end{aligned}$$

So

$$\sum_{n \leq x, \omega(n) > 10Q} n^{-1/\omega(n)} \ll xe^{-4L},$$

and obviously

$$\sum_{B < n \leq x, \omega(n) \leq 0.1Q} n^{-1/\omega(n)} \ll xe^{-4L}.$$

From the above two estimates we can get

$$\begin{aligned} (3.1) \quad T(x) &:= \sum_{2 \leq n \leq x} n^{-1\omega(n)} = \sum_{B < n \leq x, 0.1Q < k \leq 10Q} n^{-1/k} + O(xe^{-4L}) \\ &= \sum_{0.1Q < k \leq 10Q} \exp\{-(1/k) \log x + O(\log_2 x)\} (\pi(x, k) + O(xe^{-2L})) \end{aligned}$$

For $0.1Q < k \leq 10Q$, using Lemma 2 and Stirling's formula we get

$$(3.2) \quad \pi(x, k) = x \exp\{-k(\log k - 1 - \log U - (\log U/U) + O(1/U))\}.$$

From this and (3.1) we deduce

$$(3.3) \quad T(x) = x \sum_{0.1Q < k \leq 10Q} \exp\{-(1/k) \log x - k(\log k - 1 - \log U - (\log U/U) + O(1/U))\}.$$

Now we prove the upper bound

$$(3.4) \quad T(x) \leq x \exp \left\{ -2(\log x \log_2 x)^{1/2} \left(1 - \frac{3 \log_3 x}{2 \log_2 x} + \frac{3 \log 2 - 2}{2 \log_2 x} - \frac{\log_3 x}{\log_2^2 x} + O\left(\frac{1}{\log_2^2 x}\right) \right) \right\}.$$

Let

$$F(x, k) = k^{-1} \log x + k(\log k - 1 - \log U - (\log U/U) + O(1/U)),$$

and $a_i = (t + \delta_i D(\log_3 x / \log_2 x))Q$, ($i = 1, 2$), where $\delta_1 = -1$, $\delta_2 = 1$ and D denotes positive absolute constants, not necessarily the same ones. For $k \in [a_1, a_2]$ we have

$$F(x, k) = ((t/2) + (1/t))(\log x \log_2 x)^{1/2} + O(Q \log_3 x) = G(x, t), \text{ say.}$$

Hence we have

$$\begin{aligned} T(x) &\leq \max_{0.1 \leq t \leq 10} x \exp\{-g(x, t)\} \sum_{0.1Q \leq k \leq 10Q} 1 \\ &\leq x \exp\{-(2 \log x \log_2 x)^{1/2} + O((\log x)^{1/2} (\log_2 x)^{-1/2} \log_3 x)\}. \end{aligned}$$

Further let $b_i = (2^{1/2} + t_1(\log_3 x / \log_2 x) + \delta_i D(1/\log_2 x))Q$, ($i = 1, 2$).

For $k \in [b_1, b_2]$ we have

$$\begin{aligned} (3.5) \quad F(x, k) &= \left(\frac{1}{2} \log x \log_2 x\right)^{1/2} \left\{ \left(1 + \frac{t_1 \log_3 x}{2^{1/2} \log_2 x} + O\left(\frac{1}{\log_2 x}\right)\right)^{-1} + \right. \\ &\quad \left. \left(1 + \frac{t_1 \log_3 x}{2^{1/2} \log_2 x} - \frac{3 \log_3 x}{\log_2 x} + O\left(\frac{1}{\log_2 x}\right)\right) \right\} = G_1(x, t_1), \text{ say.} \end{aligned}$$

Hence

$$(3.6) \quad T(x) \leq \max_{-D \leq t_1 \leq D} x \exp\{-G_1(x, t_1)\} \sum_{0.1Q \leq k \leq 10Q} 1.$$

Let

$$f(t_1) = \left(1 + \frac{t_1 \log_3 x}{2^{1/2} \log_2 x}\right)^{-1} + \frac{t_1 \log_3 x}{2^{1/2} \log_2 x};$$

then

$$f'(t_1) = -\left(1 + \frac{t_1 \log_3 x}{2^{1/2} \log_2 x}\right)^{-2} \frac{\log_3 x}{2^{1/2} \log_2 x} + \frac{\log_3 x}{2^{1/2} \log_2 x},$$

which means that $f'(t_1) = 0$ for $t_1 = 0$. From this and (3.5), (3.6) we have

$$T(x) \leq x \exp\left\{-(2 \log x \log_2 x)^{1/2} \left(1 - \frac{3 \log_3 x}{2 \log_2 x} + O\left(\frac{1}{\log_2 x}\right)\right)\right\}.$$

Proceeding as before, finally let

$$e_1 = (2^{1/2} + t_4(\log_3 x / \log_2^2 x) + \delta_i D(1/\log_2^2 x))Q, \quad (i = 1, 2).$$

For $k \in [e_1, e_2]$ we have

$$\begin{aligned} (3.7) \quad F(x, k) &= \left(\frac{1}{2} \log x \log_2 x\right)^{1/2} \left\{ \left(1 + \frac{t_4 \log_3 x}{2^{1/2} \log_2^2 x} + O\left(\frac{1}{\log_2^2 x}\right)\right)^{-1} + \right. \\ &\quad \left. \left(1 - \frac{3 \log_3 x}{\log_2 x} + \frac{3 \log 2 - 2}{\log_2 x} + \left(\frac{t_4}{2^{1/2}} - 2\right) \frac{\log_3 x}{\log_2 x} + O\left(\frac{1}{\log_2^2 x}\right)\right) \right\} \\ &= G_4(x, t_4), \text{ say.} \end{aligned}$$

Hence we have

$$(3.8) \quad T(x) \leq \max_{-D \leq t_4 \leq D} x \exp\{-G_4(x, t_4)\} \sum_{0.1Q \leq k \leq 10Q} 1.$$

Let

$$g(t_4) = \left(1 + \frac{t_4 \log_3 x}{2^{1/2} \log_2^2 x}\right)^{-1} + \frac{t_4 \log_3 x}{2^{1/2} \log_2^2 x};$$

then

$$g'(t_4) = -\left(1 + \frac{t_4 \log_3 x}{2^{1/2} \log_2^2 x}\right)^{-2} + \frac{\log_3 x}{2^{1/2} \log_2^2 x} + \frac{\log_3 x}{2^{1/2} \log_2^2 x},$$

which means that $g'(t_4) = 0$, for $t_4 = 0$. From this and (3.7), (3.8), we can derive (3.4).

To finish the proof of Theorem 2 it remains to show

$$(3.9) \quad T(x) \geq x \exp\left\{- (2 \log x \log_2 x)^{1/2} \left(1 - \frac{3 \log_3 x}{2 \log_2 z} + \frac{3 \log 2 - 2}{2 \log_2 x} - \frac{\log_3 x}{\log_2^2 x} + O\left(\frac{1}{\log_2^2 x}\right)\right)\right\}.$$

Let $k_0 \leq (2 \log x / \log_2 x)^{1/2}$. It is evident that

$$T(x) \geq \sum_{2 \leq n \leq x, \omega(n)=k_0} n^{-1/\omega(n)} \geq x^{-1/k_0} \pi(x, k_0).$$

From this and (3.2) we have

$$(3.10) \quad T(x) \geq x \exp\left\{- (1/k_0) \log x - k_0 (\log k_0 - 1 - \log U - (\log U/U) + O(1/U))\right\}$$

Now putting $k_0 = [(2 \log x / \log_2 x)^{1/2}]$ in (3.10), we can derive (3.9). The proof is completed.

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