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ON DINSTANCES IN SOME BIPARTITE GRAPHS

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Abstract. Let $d(v \mid G)$ be the sum of the dinstances between a vertex v of a graph G and all other vertices of G. Let $W(G)$ be the sum of the distances between all pairs of vertices of G. A class $\mathbf{C}k$ of bipartite graphs is found, such that $d(v \mid G) \equiv 1 \pmod k$ holds for an arbitrary vertex of an arbitrary member of $\mathbf{C}(k)$. Further, for two members G and H of $\mathbf{C}(k)$, having equal cyclomatic number, $W(G) \equiv W(H) \pmod{2k^2}$.

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Introduction

In the present paper we establish certain properties of the vertex distances of some bipartite graphs. If G is a (connected) graph and u and v are its two vertices, then the length of the shortest path which connects u and v is denoted by $d(u, v)$ and is called the distance between u and v. The sum of the distances between the vertex v and all other vertices of G is denoted by $d(G \mid v)$. The sum of the distances between all pairs of vertices of G is denoted by $W(G)$ og simply by W. Hence,

$$W = W(G) = \sum_{\{u,v\}} d(u, v)$$

where $\{u, v\}$ runs over all two-elemnt subsets of the vertex set of G.

We mention in passing that the quantity W plays some role in chemistry [1]. In the chemical literature $W(G)$ is called the Wiener number of the graph G.

Let G be a connected bipatite graph and X and Y its two pertient vertex sets. Then one immediately sees that $d(u, v)$ is even if both u nad v belong to either X or to Y. Otherwise, $d(u, v)$ is odd. This implies the following consequence.

LEMMA 1. *$d(v \mid G) \equiv 1$ (mod 2 iff either $v \in X$ and $\mid Y \mid$ is odd or $v \in Y$ and $\mid X \mid$ is odd. Further, $W(G) \equiv 1$ (mod 2) iff both $\mid X \mid$ and $\mid Y \mid$ are odd.*

In the present paper we prove a number of additional congruence statements for the numbers $d(v \mid G)$ and $W(G)$, which hold for the elements of the sets $\mathbf{C}(h, k)$ and $\mathbf{C}(k)$.

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Definition. Let k be a positive integer. If $h > 1$, then every element of $\mathbf{C}(h,k)$ is a graph optained by joining the endpoints of a path with $2k$ vertices to a pair of adjacent vertices of some graph from $\mathbf{C}(h-1,k)$. The set $\mathbf{C}(1,k)$ consists of one element only— the circuit with $2k+2$ vertices.

It is both consistent and convenient to define $\mathbf{C}(0,k)$ as the one-element set, containing graph on two vertices.

The union of the sets $\mathbf{C}(h,k), h = 0,1,2,\ldots$ is denoted by $\mathbf{C}(k)$

For example, $\mathbf{C}(4,1)$ consists of eight elements, namely the eight graphs depicted in Fig. 1.

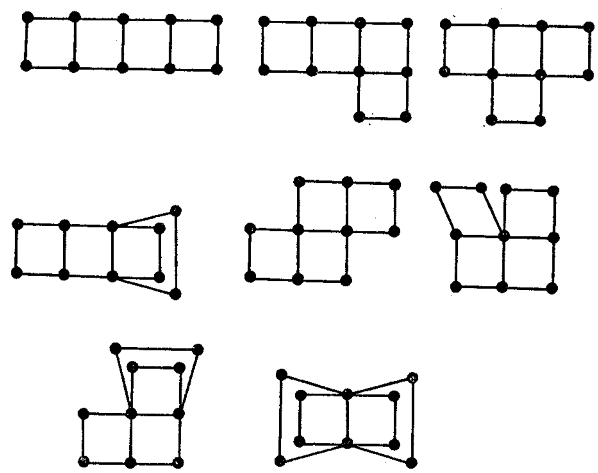

Fig. 1

The basic properites of the above defined classes of graphs are collected in the following lemma.

LEMMA 2. *If G is a graph from $\mathbf{C}(h,K)$, then*

(a) *G is a connected bipartite graph with $\mid X \mid = \mid Y \mid$;*

(b) the cyclomatic number of G is h;

(c) the girth of G is $2k+2$ and every edge of G belongs to a$(2k+2)$-membered circuit;

(d) G has $\mid G \mid = 2kh+2$ vertices.

The main results

THEOREM 1. *If graph from $\mathbf{C}(k)$ and v is its arbitrary vertex, then*

$$d(v \mid G) \equiv 1 \pmod{k}. \tag{1}$$

If, further, k is even, then

$$d(v \mid G) \equiv 1 \pmod{2k}. \tag{2}$$

THEOREM 2. *If G and H are graphs from $\mathbf{C}(h,k)$, then*

$$W(G) \equiv W(H) \ \mathrm{mod}\ 2k^2). \tag{3}$$

Proof of Theorem 1. We demonstrate the validity of Theorem 1 for $G \in \mathbf{C}(h,k)$ by induction on h. For $h = 0$, (1) and (2) hold in a trivial manner since then $W = 1$. For the unique graph from $\mathbf{C}(1,k)$, namely the circuit with $2k+2$ vertices, it is easy to show that $d(v \mid G) = (k+1)^2$. Whence (1) and (2) are satisfied.

Suppose now that G^* is an element of $\mathbf{C}(h-1,k)$ and that G can be obtained by joinning the endpoints u_1 and u_{2k} of a path with $2k$ vertices with the vertices p and q of G^* (see Fig. 2).

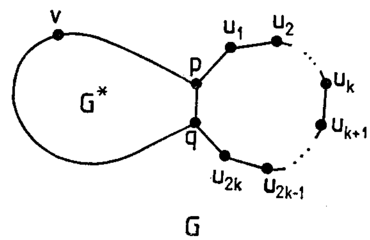

Fig. 2

The newly introduced vertices of G are labeled by $u_1, u_2, \dots, u_{2k}$.

From the construction of the graph G it is evident that

$$d(v \mid G) = d(v \mid G^*) + \sum_{i=1}^{2k} d(v, u_i).$$

Assuming that $d(v,p) < d(v,q)$, we have

$$\sum_{i=1}^{k} d(v,u_i) = kd(v,p) + k(k+1)/2,$$

$$\sum_{i=k+1}^{2k} d(v,u_i) = k + kd(v,p) + k(k+1)/2,$$

and

$$d(v \mid G) = d(v \mid G^*) + 2kd(v,p) + k(k+2).$$

Consequently,

$$d(v \mid G) \equiv d(v \mid G^*) \pmod{k}$$

and, if k is even,

$$d(v \mid G) \equiv d(v \mid G^*) \pmod{2k}.$$

Therefore if (1) and (2) hold for the vertex v of G^*, then they also hold for the vertex v of G.

In order to complete the proof of Theorem 1, we have to show that (1) nad (2) hold also for the vertices $u_i, i = 1, 2, \ldots, 2k$ of G. Let u stand for one of these vertices. Then

$$d(u \mid G) = \sum_{i=1}^{2k} d(u, u_i) + d(u, p) + d(u, q) + \sum_{v} d(u, v) \tag{4}$$

with the second summation of the r.h.s. of (4) running over the vertices of G^* different than p and q. The vertices $u_1, u_2, \ldots, u_{2k}, q, p$ form a circuit of size $2k+2$ in G and therefore

$$\sum_{i+1}^{2k} d(u, u_i) + d(u, p) + d(u, q) = k(k+1)^2.$$

Further,

$$\sum_{v} d(u, v) = d(x \mid G^*) - 1 + (\mid G^* \mid -2) d(u, x)$$

where $x = p$ if $d(u, p) < d(u, q)$ and $x = q$ otherwise. According to Lemma 2, $\mid G^* \mid = 2k(h-1) + 2$.

Taking all this into account, eq. (4) becomes

$$d(u \mid G) = d(x \mid G^*) + k(k+2) + 2k(h-1) d(u, x),$$

Hence,

$$d(u \mid G) \equiv d(x \mid G^*) \pmod{k}$$

and, if k is even,

$$d(u \mid G) \equiv d(x \mid G^*) \pmod{2k}.$$

This means that if (1) and (2) hold for all vertices of $G^* \in \mathbf{C}(h-1), k)$, then they also for all vertices of $G \in \mathbf{C}(h, k)$. Consequently, they hold for all vertices of all graphs from $\mathbf{C}(k)$. $\square$

Proof of Theorem 2. The sets $\mathbf{C}\ (0, k)$, $\mathbf{C}(1, k)$ and $\mathbf{C}(2, k)$ contain one element each and therefore for $h = 0, 1$ and 2 Theorem 2 holds in a trivial manner. Direct calculation confirms the validity of Theorem 2 also in the case of $\mathbf{C}(3, k)$ (which contains $k+2$ elements).

In order to complete a proof by induction, assume that (3) is obeyed for all graphs from $\mathbf{C}(h-1, k)$ and in particular for G^* and H^*. Let $G \in \mathbf{C}(h, k)$ be obtained from G^* in the previously described way (see Fig. 2). Let $H \in \mathbf{C}(h, k)$ be obtained from H^* in a fully analogous manner.

Now, from Fig. 2 we see that

$$W(G) = W(G^*) + k(4k^2 - 1)/3 + k(k+1) \mid G^* \mid + k[d(\mid G^*) + d(q \mid G^*)]. \quad (5)$$

Namely, $k(4k^2 - 1)/3$ is the W number of the path with $2k$ vertices, where as the sum of the distances between the vertices of G^* and $u_1, u_2, \ldots, u_k$ is

$$kd(p \mid G^*) \mid G^* \mid \sum_{i=1}^{k} i$$

and, similary, the sum of the distances between the vertices of G^* and $u_{k+1}, \ldots, {}_{2k}$ is

$$kd(q \mid G^*) + \mid G^* \mid \sum_{i=0}^{k} i.$$

An analogous equality will hold for the graph H, viz.,

$$W(H) = W(H^*) + k(4k^2 - 1)/3 + k(k+1) \mid H^* \mid +$$
$$k \mid d(p \mid H^*) + d(\mid q \mid H^*) \mid . \quad (6)$$

Bearing in mind that by Lemma 2, $\mid G^* \mid = \mid H^* \mid$, the identities (5) and (6) yield

$$W(G) - W(H) = W(G^*) - W(H^*) + k[d(p \mid G^*) - d(p \mid H^*) + d(q \mid G^*) - d(q \mid H^*)].$$

We now have to distinguish between two cases.

Case a: k is even. Then because of Theorem 1, $d(p \mid G^*) - d(p \mid H^*) + d(q \mid G^*) - d(q \mid H^*)$ is divisible by $2k$. Therefore the induction hypothesis that $W(G^*) - W(H^*)$ is divisible by $2k^2$ leads to the conclusion that then also $W(G) - W(H)$ is divisible by $2k^2$.

Theorem 2 follows for the case of even k.

Case b: k is odd. Then by Theorem 1, $d(p \mid G^*) - d(p \mid H^*) + d(q \mid G^*) - d(q \mid H^*)$ is divisible only by k and the above reasoning leads to the conclusion that $W(G) - W(H)$ is divisible by k^2. On the other hand, according to Lemma 1, $W(G) - W(H)$ must be divisible by two. Since k^2 is assumed to be odd, $W(G) - W(H)$ must be divisible by $2k^2$.

This proves Theorem 2 also for odd values of k. $\square$

Discussion

1° In the set $\mathbf{C}(3, k)$ there exist graphs G and H such that $W(G) - W(H) = 2k^2$. Therefore $2k^2$ is the greatest possible argument in a relation of the type (3) and Theorem 2, is, in a certain sense, the best possible congruence statement for the W numbers of the members of $\mathbf{C}(h, k)$.

2° Eq. (5) and the fact that $\mid G \mid = 2k(h-1) + 2$ imply

$$W(G) - W(G^*) \equiv k(k^2 + 11)/3 (\mathrm{mod}\ \ k^2).$$

This means that by increasing the cyclomatic number by one, W increases by $k(k^2+11)/3$ modulo k^2. Since for the (unique) element of $\mathbf{C}(0,k)$ it is $W=1$, one concludes that for the members of $\mathbf{C}\ (h,k)$.

$$W \equiv 1 + hk(k^2+11)/3 \pmod{k^2}.$$

Analogously, if k is even, then

$$W \equiv 1 + hk(k^2+11)/3 \pmod{2k^2}.$$

3° There is a natural way to generalize the definition of the presently considered bipartite graphs to non-bipartite ones. Instead of the set $\mathbf{C}(h,k)$ we may define that the set $\mathbf{C}^*(h,k)$ whose elements are constructed by joining the endpoints of a path with $2k-1$ vertices to a pair adjacent vertices of a graph from $\mathbf{C}^*(h-1,k)$. Further, $\mathbf{C}^*(1,k)$ would consist of the circuit of the size $2k+1$.

Unfortunately, neither Theorem 1 nor Theorem 2 could be extended to $\mathbf{C}^*(h,k)$, nor any other similar congruence statement could be established.

REFERENCES

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