PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 42 (56), 1987, pp. 155–165

A FIRST ORDER ACCURACY SCHEME ON NON-UNIFORM MESH

Mirjana Stojanović

Abstract. It is proved that the exponentially fitted quadratic spline difference scheme derived in [5] and applied to the singularly perturbed two-point boundary value problem

$$\varepsilon y'' + p(x)y' = f(x), \quad 0 < x < 1,\ 0 < \varepsilon \ll 1,$$
$$y(0) = \alpha_1,\ y(1) = \alpha_1,\ p(x) \geq p > 0.$$

has the first order of uniform convergence on non-uniform mesh. In order to achieve the uniform first order accuracy the special "almost uniform mesh" which satisfies the condition $h_i = h_{i-1} + Mh_{i-1} \max(x_i, \varepsilon)$ was constructed. The results are illustrated by numerical experiments.

Introduction. In [5] is given an exponentially fitted quadratic spline difference scheme for solving singularly perturbed two-point boundary value problem

$$(1) \quad Ly = \varepsilon y'' + p(x)y' = f(x),\ 0 < x < 1,\ 0 < \varepsilon \ll 1, \quad y(0) = \alpha_0, \quad y(1) = \alpha_1,$$

where α_0, α_1 are given constants on uniform mesh. The scheme has the uniform first order accuracy.

In this paper we introduce for the scheme derived in [5] a non-uniform mesh $0 = x_0 < x_1 < \cdots < x_n = 1$, $h_i = x_{i+1} - x_i$ in order to increase the number of points in the boundary layer.

There are two possibilities to obtain a small truncation error inside the layer, namely, to choose a fine mesh there, or-what is less trivial-to choose a difference formula reflecting the behaviour of the solution inside the layer. By chaining the fitting factor ([5]) we have not succeeded to find a difference formula which on non-uniform mesh, for $pi =$ const would give truncation error equal to zero, or small enough.

AMS Subject Classification (1980): Primary 65L10.

Key words and phrases: Exponentially fitted quadratic spline, singularly perturbed problems, uniform convergence on non-uniform mesh.

So we choose the mesh which satisfies the condition,

$$h_i = h_{i-1} + Mh_{i-1}\max(x_i, \varepsilon)$$

to preserve the same order of uniform accuracy which has been obtained on uniform mesh.

The scheme [5] has the form

$$R^h v_i = Q^h f_i, \quad i = 1(1)n-1, \quad v_0 = \alpha_0, v_n = \alpha_1, \tag{2}$$

where $R^h v_i = r_i^- v_{i-1} + r_i^c v_i + r_i^+ v_{i+1}$, $Q^h f_i = q_i^- f_{i-1} + q_i^c f_i + q_i^+ f_{i+1}$. The coefficients of the scheme (2) are

$$r_i^- = \frac{1}{h_i}\left(\frac{\operatorname{cth}\varrho_i + 1}{\operatorname{cth}\varrho_i}\right),\ r_i^- = \frac{1}{h_{i-1}}\left(\frac{\operatorname{cth}\varrho_{i-1} - 1}{\operatorname{cth}\varrho_{i-1}}\right),\ r_i^c = -r_i^- - r_i^+,$$
$$q_i^- = \frac{1}{p_{i-1}}\frac{1}{\operatorname{cth}\varrho_{i-1}},\ q_i^c = \frac{1}{p_i}\frac{1}{\operatorname{cth}\varrho_i},\ q_i^+ = 0,\ \varrho_i = \frac{p_i h_i}{2\varepsilon}$$

$p_i = p(x_i)$, $f_i = f(x_i)$, where x_i is the node-point of the subinterval $[x_i, x_{i+1})$, $i = 1(1)n-1$.

For the scheme above the following results have been obtained by using the comparison function approach. The proof was derived in a fashion analogous to the construction presented in [1] and consists of observing the two comparison function $\varphi_i = -2 + x$ and $\psi = \exp(-\beta x/\varepsilon)$, where β is a constant to be chosen. The following lemma bounds the behaviour of the solution $y(x)$ of (1) and is used in the comparison function proof to bound the truncation error.

The division into two comparison functions is a consequence of properties of the exact solution.

LEMMA 1. [1] *Let $f, p \in C^3[0,1]$. Then the solution of* (1) *has the form $y(x) = u(x) + w(x)$, where*

$$u(x) = -\varepsilon y'(0)\exp(-p(0)x/\varepsilon)/p(0) \tag{3}$$

$$|w^{(i)}(x)| \le M(1 + \varepsilon^{-i+1}\exp(-2\delta x/\varepsilon)),\ i = 0(1)n,\ |\varepsilon y'(0)| \le M, \tag{4}$$

and M and δ are constants independent of ε and h.

Thus,

$$|y(x)| \le M(\exp(-p(0)x/\varepsilon)) + |w(x)|. \tag{5}$$

Let $z_i = y(x_i) - v_i$ where v_i is the approximate and $y(x_i)$ is the exact solution of the problem (1). With τ_i is denoted the truncation error of the scheme (2). The scheme (2) satisfies the maximum principle [7].

LEMMA 2. (Maximum principle) *Let $\{v_i\}$ be a set of values at the grid points x_i satisfying $v_0 \le 0$, $v_k \le 0$, and $R^h v_i \ge 0$, $i = 1(1)n-1$. Then $v_i \le 0$, $i = 0(1)n$.*

COROLLARY 1. *If $k_1(h_i,\varepsilon) \geq 0$, $k_2(h_i,\varepsilon) \geq 0$ are such functions that $R^h(k_1\varphi_i + k_2\psi_i) > R^h(\pm z_i) = \pm\tau_i$ then $|z_i| \leq k_1|\varphi_i| + k_2|\psi_i|$.*

To carry out a comparison function proof it is necessary to find lower bounds for $R^h\varphi_i$ and $R^h\psi_i$ to bound $|\tau_i|$, thus determining k_1 and k_2 and hence giving an error estimate.

The proof of the uniform convergence on non-uniform mesh consists of Taylor's expansions of the truncation error of the scheme (2) and operators in the point (h_i, h_i). This way we separately observe the equidistant part and the part which multiplies (h_{i-1}, h_i) with derivation of order h_{i-1}. The proof for uniform part is given in [5]. Here is shown that the non-uniform parts are of lower order than uniform parts, but with the mentioned condition on the mesh. So the order of uniform convergence achieved on the uniform mesh is preserved.

LEMMA 3. *If the non-uniform mesh is regular, then there are constants M and β independent of ε an h_i, so that the following inequalities hold:*

(a) $R^h\varphi_i \geq Mh_i/\varepsilon$ *for* $h_i \leq \varepsilon$,

(b) $R^h\varphi_i \geq M$ *for* $h_i \geq \varepsilon$,

(6) (c) $R^h\psi_i \geq M(\mu_i(\beta)h_i/\varepsilon)/\varepsilon$ *for* $h_i \leq \varepsilon$,

(d) $R^h\psi_i \geq M(\mu_i(\beta))/h_i$ *for* $h_i \geq \varepsilon$,

(e) $R^h\psi_i/\exp(-\beta h_{i-1}/\varepsilon) \geq M(\mu_{i-1}(\beta))/h_i$ *for* $h_i \geq \varepsilon$

$\mu(\beta) = \exp(-\beta h_i/\varepsilon)$, *where β is a constant be to chosen; $h_i = \max(h_i, h_{i-1})$.*

Proof (a), (b) $R^h\varphi_i = -h_{i-1}r_i^- + h_i r_i^+ = (\operatorname{cth}\varrho_i)^{-1} + (\operatorname{cth}\varrho_{i-1})^{-1}$ and the estimate

$$\operatorname{cth}\varrho_{i-1} = \operatorname{cth}\varrho_i + 0(M(h_{i-1} - h_i)\varepsilon/h_i^2) \quad \text{for} \quad h_i \leq \varepsilon. \tag{7}$$

yields

$$R^h\varphi_i \geq Mh_i/\varepsilon \quad \text{for} \quad h_i \leq \varepsilon, \quad \text{and} \quad R^h\varphi_i \geq M \quad \text{for} \quad h_i \geq \varepsilon.$$

One can check that

$$\begin{aligned} R^h\psi_i = r_i^+(\exp(-\beta x_{i-1}/\varepsilon)[(r_i^-(\exp(-\beta h_{i-1}/\varepsilon) - 1))/r_i^+ + \\ + \exp(-\beta h_{i-1}/\varepsilon)(1 - \exp(-\beta h_i/\varepsilon))]. \end{aligned} \tag{8}$$

By expanding this operator into Taylor's series we obtain

$$\begin{aligned} R^h\psi_i(h_i, h_{i-1}) &= R^h\psi_i(h_i, h_i) + (h_{i-1} - h_i)\frac{dR^h\psi_i}{dh_{i-1}}(h_i, h_i) \\ &+ \frac{1}{2}(h_{i-1} - h_i)^2\frac{d^2R^h\psi_i}{dh_{i-1}^2}(h_i, h_i + \theta(h_{i-1} - h_i)), \quad 0 < \theta < 1. \end{aligned}$$

$R^h\psi_i(h_i, h_i)$ is estimated in [5]. These estimates are the same as in Lemma 3 when $h_i = h$.

The estimates (c), (d) are obtained by estimating the individual factors in (8) for the three cases

(i) $h_i/\varepsilon \le c$, (ii) $h_i/\varepsilon \ge C$, and (iii) $c \le h_i/\varepsilon \le C$, (For appropriately chosen c and C.)

For (i) and c sufficiently small

$$\frac{dR^h\psi_i}{dh_{i-1}}(h_i,h_i) = \mu_i(\beta)(-\beta/\varepsilon h_i)(-\rho_i - (\rho_i/\operatorname{cth}\rho_i) + M(\beta h_i/\varepsilon)). \tag{9}$$

This implies

$$\left|\frac{dR^h\psi_i}{dh_{i-1}}(h_i,h_i)\right| \ge M_2\mu_i(\beta)/\varepsilon^2.$$

Thus,

$$|R^h\psi_i(h_i,h_i)| \ge Mh_i\mu_i(\beta)/\varepsilon^2 + (h_{i-1}-h_i)\cdot M_1\mu_i(\beta)/\varepsilon^2, \text{ for } h_i \le c\varepsilon.$$

With our condition on the mesh this is obvious, but in fact this estimate does not require any condition on the mesh. (All estimates are given for $\operatorname{cth}\rho_i$, and they are valid for $\operatorname{cth}\rho_{i-1}$ because of (7)). The following estimate also holds

$$\rho_{i-1} = \rho_i + 0(M(h_{i-1}-h_i)/\varepsilon). \tag{10}$$

For (ii) and for C sufficiently large

$$1-\exp(-\beta h_i/\varepsilon) \ge M_1, \quad 1-\exp(-\beta h_{i-1}/\varepsilon) \ge M_2,$$

$$|r_i^+| \ge M_3/h_i, \quad \frac{r_i^-}{r_i^+} = \frac{h_i}{h_{i-1}}\cdot\frac{2\exp(-2\rho_{i-1})}{1-\exp(-2\rho_{i-1})}\cdot\frac{\operatorname{cth}\rho_i}{\operatorname{cth}\rho_{i-1}(\operatorname{cth}\rho_i+1)}.$$

If $\varepsilon \to 0$, then $\operatorname{cth}\rho_i/(\operatorname{cth}\rho_{i-1}(\operatorname{cth}\rho_i+1)) \to 1/2$. Then the expression in brackets in (8) is bounded by $M_4\exp(-\beta h_{i-1}/\varepsilon)$. So we have (d):

$$|R^h\psi_i| > M_5(\mu_i(\beta))h_i.$$

(iii) (c and C are now fixed) and for h_i sufficiently small

$$r_i^-/r_i^+ = \exp(-2\rho_{i-1}) + 0(M_6), \quad r_i^+ \ge M_7/h_i, \quad 1-r_i(\beta) \ge M_8,$$

and we obtain $|R^h\psi_i| > M_9(\mu_i(\beta))h_i$. The expression (e) we obtain by dividing (d) by $\exp(-\beta h_{i-1}/\varepsilon)$.

Theorem 1. *Let $f,p \in C^3[0,1]$. Let $\{v_i\}$, $i = 0(1)n$ be a set of values of the approximate solution of problem* (1) *obtained by using scheme* (2)*, on the mesh which satisfies the condition $h_i = h_{i-1} + Mh_{i-1}\max(x_i,\varepsilon)$. Then the following estimate holds:*

$$\begin{cases} |y(x_i)-v_i| \le Mh_i(1+\exp(-\delta x_i/\varepsilon)) \text{ for } h_i \le \varepsilon \\ |y(x_i)-v_i| \le Mh_i(1+\exp(-\delta x_{i-1}/\varepsilon)) \text{ for } h_i \ge \varepsilon. \end{cases} \tag{11}$$

Proof. The truncation error of y is the sum of the truncation errors of functions u an w

$$\tau_i(y) = \tau_i(u) + \tau_i(w).$$

We separately give the proof for functions $w(x)$ and $u(x)$. We start with $w(x)$. The truncation error is defined as

$$\tau_i = R^h y(x_i) - Q(L(y(x_i)) \text{ for } i = 1(1)n-1.$$

For $y(x)$ sufficiently smooth the standard Taylor expansion of τ_i for ε fixed has the form (see [1]):

$$\tau_i(y) = \tau_i^0 y(x_i) + \tau_i^1 y'(x_i) + \tau_i^2 y''(x_i) + \cdots + R.$$

One can easily verify that, for the scheme (2) τ_i^0 and τ_i^1 are equal to zero, but we must estimate τ_i^2.

$$\tau_i^2 = \frac{h_i}{2}\left(1 + \frac{1}{\operatorname{cth}\varrho_i}\right) + \frac{h_{i-1}}{2}\left(1 - \frac{1}{\operatorname{cth}\varrho_{i-1}}\right) - \frac{\varepsilon}{p_{i-1}\operatorname{cth}\varrho_{i-1}} - \frac{\varepsilon}{p_i \operatorname{cth}\varrho_i} + \frac{h_{i-1}}{\operatorname{cth}\varrho_{i-1}}$$

After ordering these terms we obtain

$$\tau_i^{(2)} = \frac{(h_i p_i \operatorname{cth}\varrho_i - 2\varepsilon)}{2p_i \operatorname{cth}\varrho_i} + \frac{(h_{i-1}p_{i-1}\operatorname{cth}\varrho_{i-1} - 2\varepsilon)}{2p_{i-1}\operatorname{cth}\varrho_{i-1}} + \frac{h_i}{2}\frac{1}{\operatorname{cth}\varrho_i} + \frac{h_{i-1}}{2}\frac{1}{\operatorname{cth}\varrho_{i-1}}$$
$$\tau_i^{(2)} = \frac{w_i}{2p_i\operatorname{cth}\varrho_i} + \frac{w_{i-1}}{2p_{i-1}\operatorname{cth}\varrho_{i-1}} + \frac{h_i}{2}\frac{1}{\operatorname{cth}\varrho_i} + \frac{h_{i-1}}{2}\frac{1}{\operatorname{cth}\varrho_{i-1}}$$

For w_i we can estimate $|w_i| \le h_i^2/(h_i+\varepsilon)$ (see [4]). And we obtain

$$|\tau_i^2| \le Mh_i^2/\varepsilon \text{ for } h_i \le \varepsilon, \quad |\tau_i^{(2)}| \le Mh_i \text{ for } h_i \ge \varepsilon.$$

This implies

$$|\tau_i^{(2)}(w_i)| \le Mh_i^2(1 + \exp(-\delta x_i/\varepsilon)/\varepsilon)/\varepsilon \text{ for } h_i \le \varepsilon$$

and

$$|\tau_i^{(2)}(w_i)| \le Mh_i(1 + \exp(-\delta x_i/\varepsilon)/\varepsilon) \text{ for } h_i \ge \varepsilon$$

Now we must estimate $\tau_i^{(3)}$

$$\tau_i^{(3)} = \frac{h_i}{6}\left(1 + \frac{1}{\operatorname{cth}\varrho_i}\right) - \frac{h_{i-1}}{6}\left(1 - \frac{1}{\operatorname{cth}\varrho_{i-1}}\right) + \frac{\varepsilon h_{i-1}}{p_{i-1}}\frac{1}{\operatorname{cth}\varrho_{i-1}} - \frac{h_{i-1}^2}{2}\frac{1}{\operatorname{cth}\varrho_{i-1}}$$

and we have

$$|\tau_i^{(3)}| \le Mh_i^2 \text{ for } h_i \le \varepsilon \text{ and } |\tau_i^{(3)}| \le Mh_i^2 \text{ for } h_i \ge \varepsilon.$$

Thus,

$$|\tau_i^{(3)}(w_i)| \le Mh_i^2(1 + \exp(-\delta x_i/\varepsilon)/\varepsilon^2) \text{ for all } h_i \text{ and } \varepsilon.$$

We can prove that the non-elaborated parts of $\tau_i(y)$ are of a minor order. In a similar way we can estimate the remainder terms.

Applying Lemma 3 and Corollary I we obtain the final estimate for $w(x)$.

Estimate for $u(x) = \exp(-p(0)x/\varepsilon)$. The proof is based on the relation $\tau_i(y) = R^h y_i - Q^h Ly = R^h u_i - R^h v_i = \tau_r - \tau_q$. (See [1]). We first consider the case when $h_i \le \varepsilon$.

Expanding this truncation error in Taylor's series at the point (h_i, h_i) we obtain

$$\tau(h_i, h_{i-1}) = \tau(h_i, h_i) + (h_{i-1} - h_i)\frac{d\tau_i}{dh_{i-1}}(h_i, h_i) + \\ + \frac{(h_{i-1} - h_i)^2}{2}\frac{d^2\tau_i}{dh_{i-1}^2}(h_i, h_i + \theta(h_{i-1} - h_i)), \quad 0 < \theta < 1.$$

The part $\tau_i(h_i, h_i) = \tau_r(h_i, h_i) + \tau_q(h_i, h_i)$ is defined on uniform mesh and is estimated in [5]. This estimate is the same as (11), when $h_i = h$.

Now we can estimate the first derivation of τ_i on the power h_{i-1}.

$$\frac{d\tau_i}{dh_{i-1}}(h_i, h_i) = \frac{d\tau_r}{dh_{i-1}}(h_i, h_i) - \frac{d\tau_q}{dh_{i-1}}(h_i, h_i).$$

We shall estimate these parts separately.

$$\tau_r(h_i, h_{i-1}) = v_i^* \frac{1}{h_{i-1}}\left(1 - \frac{1}{\operatorname{cth}\rho_{i-1}}\right)\left(\exp\left(\frac{p_0}{\varepsilon}h_{i-1}\right) - 1\right).$$

where $v_i^* = \exp(-p_0 x_i/\varepsilon)$.

Throughout the rest of the paper the terms which have been reduced to a nice form where (6) yields (11) will be generically denoted by N.

$$\begin{aligned}
&\frac{d\tau_r}{dh_{i-1}}(h_i, h_i) = \\
&= \frac{v_i^*}{h_i^2}\frac{(\operatorname{cth}\rho_{i-1} - 1)}{\operatorname{cth}\rho_{i-1}}\left[\left(\exp\left(\frac{p_0}{\varepsilon}h_i\right) - 1\right)\left(-\rho_{i-1} - \frac{\rho_{i-1}}{\operatorname{cth}\rho_{i-1}} - 1\right) + \right.\\
&\left. + \left(\frac{p_0}{\varepsilon}h_i\right)\exp\left(\frac{p_0}{\varepsilon}h_i\right)\right] = \frac{v_i^*}{h_i}\frac{p_0}{\varepsilon}\frac{(\operatorname{cth}\rho_{i-1} - 1)}{\operatorname{cth}\rho_{i-1}}\left\{\left[1 + \frac{1}{2}\left(\frac{p_0}{\varepsilon}h_i\right) + N\right]\right.\\
&\quad\left.\left[-\rho_{i-1} - \frac{\rho_{i-1}}{\operatorname{cth}\rho_{i-1}} - 1\right] + \left[1 + \left(\frac{p_0}{\varepsilon}h_i\right) + \frac{1}{2}\left(\frac{p_0}{\varepsilon}h_i\right)^2 + N\right]\right\} = \\
&= \frac{v_i^*}{h_i}\frac{p_0}{\varepsilon}\frac{(\operatorname{cth}\rho_{i-1} - 1)}{\operatorname{cth}\rho_{i-1}}\left\{\frac{1}{2}\frac{(p_0 - p_{i-1})}{\varepsilon}h_i + O\left(\frac{h_i^2}{\varepsilon^2}\right)\right\}.
\end{aligned}$$

Now we estimate the part τ_q.

$$-\tau_q(h_i,h_{i-1}) = -\tau_q(h_i,h_i) + (h_{i-1}-h_i)\frac{d\tau_q}{dh_{i-1}}(h_i,h_i) +$$

$$+\frac{1}{2}(h_{i-1}-h_i)^2\frac{d^2\tau_q}{dh_{i-1}^2}(h_i,h_i+\theta_1(h_{i-1}-h_i)), \quad 0<\theta<1,$$

(14)
$$-\tau_q(h_i,h_{i-1}) = v_i^*\frac{p_0}{\varepsilon}\cdot\frac{(p_0-p_{i-1})}{p_{i-1}}\cdot\frac{\exp(p_0h_{i-1}/\varepsilon)}{\operatorname{cth}\rho_{i-1}}$$

$$\frac{d\tau_q}{dh_{i-1}}(h_i,h_i) = \frac{v_i^*}{h_i}\left(\frac{p_0}{\varepsilon}\right)\frac{(p_0-p_{i-1})}{p_{i-1}}\frac{\exp(p_0h_i/\varepsilon)}{\operatorname{cth}^2\rho_{i-1}}\cdot$$

$$\cdot((p_0h_i\operatorname{cth}\varrho_{i-1})/\varepsilon + \varrho_{i-1}\operatorname{cth}^2\varrho_{i-1} - \varrho_{i-1}).$$

The difference between (13) and (14) gives

$$\frac{d\tau_r}{dh_{i-1}} - \frac{d\tau_q}{dh_{i-1}} = \frac{v_i^*}{h_i}\left(\frac{p_0}{\varepsilon}\right)\frac{(p_0-p_{i-1})}{p_{i-1}}\cdot\frac{1}{\operatorname{cth}^2\varrho_{i-1}}\cdot$$

$$\cdot\Bigg[\varrho_{i-1}\operatorname{cth}\varrho_{i-1}(\operatorname{cth}\varrho_{i-1}-1) - \exp\left(\frac{p_0}{\varepsilon}h_i\right)\cdot$$

$$\cdot(\varrho_{i-1}\operatorname{cth}^2\varrho_{i-1} - \varrho_{i-1} + \left(\frac{p}{\varepsilon}h_i\right)\operatorname{cth}\varrho_{i-1}\Bigg] +$$

$$+\frac{v_i^*}{h_i}\left(\frac{p_0}{\varepsilon}\right)\frac{(\operatorname{cth}\varrho_{i-1}-1)}{\operatorname{cth}\varrho_{i-1}}\cdot O(h_i^2/\varepsilon^2) =$$

$$= \frac{v_i^*}{h_i}\frac{p_0}{\varepsilon}\frac{(p_0-p_{i-1})}{p_{i-1}}\frac{1}{\operatorname{cth}^2\varrho_{i-1}}\Bigg[-\varrho_{i-1}\operatorname{cth}_{i-1} + \varrho_{i-1} -$$

$$-\left(\frac{p_0}{\varepsilon}h_i\right)\operatorname{cth}\varrho_{i-1} - \left(\frac{p_0}{\varepsilon}h_i\right)\left(1+\frac{1}{2}\left(\frac{p_0}{\varepsilon}h_i\right)+N\right)\cdot$$

$$\cdot\left(\varrho_{i-1}\operatorname{cth}^2\varrho_{i-1} - \varrho_{i-1} + \left(\frac{p_0}{\varepsilon}h_i\right)\operatorname{cth}\varrho_{i-1}\right)\Bigg] + U_M =$$

$$= \frac{v_i^*}{h_i}\cdot\frac{p_0}{\varepsilon}\cdot\frac{(p_0-p_{i-1})}{p_{i-1}}\cdot\frac{1}{\operatorname{cth}^2\varrho_{i-1}}[O(M)] + U_M.$$

The hardest parts of (13) and (14) are cacelled. Thus,

$$\left|\frac{d\tau_r}{dh_{i-1}} - \frac{d\tau_q}{dh_{i-1}}\right| \le M\frac{h_i^2}{\varepsilon}v_i^*, \quad \text{for} \quad h_i\le\varepsilon.$$

The part U_M is a part of $\dfrac{d\tau_r}{dh_{i-1}}$ which gives a condition on the mesh.

$$U_M = \frac{v_i^*}{h_i}\left(\frac{p_0}{\varepsilon}\right)\frac{\operatorname{cth}\varrho_{i-1}-1}{\operatorname{cth}\varrho_{i-1}}\cdot O\left(\frac{h_i^2}{\varepsilon^2}\right) =$$

$$|U_M| \le Mh_iv_i^*/\varepsilon^3 \quad \text{for} \quad h_i\le\varepsilon.$$

If we substitute these items in (12) we obtain:

$$(15)\qquad |\tau_i(h_i, h_{h-1})| \le M h_i^2 v_i^*/\varepsilon^2 + (h_{i-1} - h_i)[M h_i^2/\varepsilon + M h_i/\varepsilon^3] v_i^* + N.$$

We must set a condition on the mesh in order to achive the first order of uniform convergence.

If we set the condition $h_i - h_{i-1} = M_1 h_{i-1} \max(x_i, \varepsilon)$ the estimate (15) yields

$$\begin{aligned}|\tau_i(h_i, h_{h-1})| &\le M h_i^2 v_i^*/\varepsilon^2 - M_1 h_i x_i \,[M h_i^2/\varepsilon + M h_i/\varepsilon^3] v_i^* + N\\ &\le (M h_i^2/\varepsilon^2 + M_3 h_i^3 + M_3 h_i^2/\varepsilon^2) v_i^* \exp(-\delta x_i/\varepsilon) + N.\end{aligned}$$

The condition on the mesh is important because the truncation error of the scheme for uniform mesh is equal to zero for $p_i = p =$ const. and $r_i^-/r_i^+ = \exp(-2\rho_{i-1})$. But in the case of a non-uniform mesh when we separate the uniform part we obtain the part in τ_r which requires a condition on the mesh.

The final estimate of τ_i for $u(x)$ is

$$|\tau_i| \le M h_i^2/\varepsilon^2 \exp(-\delta x_i/\varepsilon), \quad 0 < \delta < 1 \quad \text{for} \quad h_i \le \varepsilon$$

Applying Lemma 3 and the maximum principle we obtain the Theorem.

We now commence the task of treating the error in $u(x)$ when $h_i \ge \varepsilon$. Then, $-\tau_q = (p_0 v_i^*/\varepsilon)[-x_{i-1} p_1'(\xi_1) q_i^- \exp(p_0 h_{i-1}/\varepsilon) - x_i p_2'(\xi_2) q_i^c]$, where $x_0 \le \xi_1 \le x_{i-1}$, and $x_0 \le \xi_2 \le x_i$. We obtain the estimate $|\tau_q| \le M \exp(-\beta x_{i-1}/\varepsilon)$ because of $|q_i^-| \le M$, and $|q_i^c| \le M$. The maximum principle and Lemma 3 leads to (11).

Taylor's expansion of $\tau_r(h_i, h_{i-1})$ at the point (h_i, h_i) is

$$\begin{aligned}\tau_r(h_i, h_{i-1}) = \tau_r(h_i, h_i) &+ \frac{h_{i-1} - h_i}{1!} \frac{d\tau_r}{dh_{i-1}}(h_i, h_i) +\\ &+ \frac{(h_{i-1} - h_i)^2}{2!} \frac{d^2\tau_r}{dh_{i-1}^2}(h_i, h_i + \theta_3(h_{i-1} - h_i)), \quad 0 < \theta_3 < 1.\end{aligned}$$

The part $\tau_r(h_i, h_i)$ is estimated in [5]. Using (13) we obtain

$$|d\tau_r/dh_{i-1}(h_i, h_i)| \le M(1/\varepsilon)(\exp(-\delta x_i/\varepsilon)).$$

The above estimate and (16) give that assertion (11) is valid for $h_i \ge \varepsilon$.

Remark 1. Using the estimate $t^k \exp(-t) \le c(\theta)\exp(-\theta t)$, $0 < t < \infty$, $0 < \theta < 1$ (see [2]) we can obtain for $h_i \ge \varepsilon$ the final estimate

$$|y(x_i) - v_i| \le M h_i + M\varepsilon \exp(-\delta x_{i-1}/\varepsilon).$$

Numerical results. The computations reported in this section were done on Delta 340 (PDP-11/34) computer in fortran IV-plus in double precision with 16 significant figures.

We give some numerical, results for our difference scheme (2) applied to the problem given in [3]: $\varepsilon y" - y' = x$, with boundary condition $y(0) = 0$, $y(1) = 0$. This problem has the exact solution:

$$y(x) = 0.5x^2 - \varepsilon x + ((\varepsilon - 0.5)/(1 - \exp(-1/\varepsilon))(1 - \exp(-x/\varepsilon)).$$

The test of uniform convergence used in this section was described in [6]. The notation is also taken from [6].

The coarsest mesh consists of 8 or 16 points arranged in interval $[0,1]$ by $h_i = h_{i-1} + Mh_{i-1}x_i$ where h_0 and M are constants to be chosen. The finer meshes are obtained by halving the previous meshes.

Table 1 indicates that the scheme (2) on a non-uniform mesh produces a first order of uniform accuracy. If we choose $h_0 = 0.05570$ and $M = 0.04481$ we obtain a mesh which consists of 5 from 16 points arranged at the first quarter of interval $[0,1]$.

$$h_0 = 0.05570, M = 0.04481.$$

$\varepsilon \backslash k$	1	2	3	4	5	P_y
1	0.101E+00	0.100E+00	0.100E+01	0.100E+01	0.100E+01	0.100E+00
1/2	0.102E+01	0.102E+01	0.101E+01	0.100E+01	0.100E+01	0.101E+01
1/4	0.104E+01	0.102E+01	0.101E+01	0.101E+01	0.100E+01	0.102E+01
1/8	0.108E+01	0.104E+01	0.102E+01	0.101E+01	0.101E+01	0.103E+01
1/16	0.115E+00	0.108E+01	0.104E+01	0.102E+01	0.101E+01	0.106E+01
1/32	0.123E+00	0.145E+01	0.108E+01	0.104E+01	0.102E+01	0.164E+01
1/64	0.125E+00	0.122E+00	0.115E+01	0.108E+01	0.105E+01	0.115E+01
1/128	0.114E+00	0.125E+00	0.123E+01	0.115E+01	0.108E+01	0.117E+01
1/256	0.967E+00	0.115E+00	0.126E+00	0.223E+01	0.115E+01	0.115E+01
1/512	0.950E+00	0.100E+00	0.115E+00	0.127E+00	0.124E+01	0.112E+00

Theoretical order of uniform convergence: 1.

Computed order of uniform convergence: 0.112E+01.

Table 2 contains the same test of uniform convergence for the special mesh which consists of 7 points at the first quarter of the interval $[0,1]$, and only one on the rest of it.

$$h_0 = 0.0569, M = 22.0000.$$

Theoretical order of uniform convergence: 1.

Computed order of uniform convergence: 0.912E+00.

$\varepsilon\backslash k$	1	2	3	4	5	P_y
1	0.957E+00	0.980E+00	0.100E+01	0.100E+01	0.100E+01	0.987E+00
1/2	0.918E+01	0.105E+01	0.103E+01	0.101E+01	0.101E+01	0.100E+01
1/4	0.864E+01	0.112E+01	0.106E+01	0.104E+01	0.102E+01	0.100E+01
1/8	0.104E+01	0.120E+01	0.113E+01	0.109E+01	0.104E+01	0.110E+01
1/16	0.906E+00	0.120E+01	0.121E+01	0.114E+01	0.108E+01	0.111E+01
1/32	0.760E+00	0.109E+01	0.123E+01	0.119E+01	0.117E+01	0.109E+01
1/64	0.715E+00	0.935E+00	0.116E+01	0.125E+01	0.119E+01	0.105E+01
1/128	0.714E+00	0.882E+00	0.100E+01	0.119E+01	0.126E+01	0.101E+01
1/256	0.710E+00	0.878E+00	0.949E+00	0.104E+01	0.120E+01	0.955E+01
1/512	0.709E+00	0.875E+00	0.945E+00	0.980E+00	0.105E+01	0.912E+00

The quotient of the difference of approximate solutions calculated at the same points of the two consequent meshes ($\log_2$ of that quotient gives the rate of uniform convergence) is smaller in the boundary layer than at the middle, or at 3/4 of the interval $[0, 1]$. Our fortran programm chooses the maximum of that difference, and in this case it tends to the end of the interval. Thus, we obtain weaker results in the Table 2, because the maximum is calculated on the part of interval which has small number of points.

We can see it from the next Table 3. If we compute for $\varepsilon = 1/64$ the difference between real and approximate solution at the second point of the meshes which have N subintervals we obtain:

$$h_0 = 0.0569, \quad M = 22.0000$$

N	32	64	128	256	512	1024
	0.211E-01	0.102E-01	0.450E-02	0.196E-02	0.888E-03	0.420E-03

The maximum error for the same ε is on the 3/4 of interval $[0, 1]$, and amounts:

N	32	64	128	256	512	1024
	0.125E-00	0.606E-01	0.268E-01	0.117E-01	0.530E-02	0.251E-02

We can compare these results with results obtained on the mesh computed in Table l. when $h_0 = 0.05570$, $N = 0.04481$, for the same ε.

The difference between exact and approximate solution in the second point is:

N	32	64	128	256	512	1024
	0.167E-01	0.740E-02	0.344E-01	0.165E-02	0.810E-02	0.401E-03

The maximum error at the third point is given by:

N	32	64	128	256	512	1024
	0.191E-01	0.846E-02	0.393E-02	0.189E-02	0.923E-03	0.458E-03

Table 4. presents some values $z_s = \max_i |y(x_i) - v_i|$ where $y(x)$ is exact and v_i is approximate solution for these meshes from Table 1 and Table 2, calculated for some ε.

$$h_0 = -0.05570, M = 0.04481.$$

$\varepsilon\backslash N$	32	64	128	256	512	1024
1/512	0.287E-01	0.135E-01	0.593E-02	0.254E-02	0.112E-02	0.522E-03
0.00001	0.306E-01	0.154E-01	0.768E-02	0.384E-02	0.191E-02	0.952E-03

$$h_0 = 0.0569, M = 22.0000$$

$\varepsilon\backslash N$	32	64	128	256	512	1024
1/512	0.137E+00	0.715E-01	0.359E-01	0.173E-01	0.799E-02	0.346E-02
0.00001	0.138E+00	0.734E-01	0.377E-01	0.191E-01	0.961E-02	0.482E-02

These results and discussion of the Table 3 suggest that it would be the best to choose a mesh with more points in the boundary layer and after that the uniform mesh (or some another non-uniform mesh because that part of solution is independent of condition on the mesh which we set).

REFERENCES

[1] A. E. Berger, J. M. and M. Ciment, *An analysis of a uniformly accurate difference method for a singular perturbation problem*, Math. Comput. **37** (1981), 79–94.

[2] E. P. Dollan, J. J. H. Miller, W. H. A. Schilders, *Uniform numerical methods for problems with initial and boundary layers*, Boole Press, Dublin, 1980, 223–238.

[3] A. M. Il'in, *Raznostnaja shema dlja differencialnogo upravnenija s malym parametrom pri staršej proizvodnoj*, Mat. Zametki **6** (1969), 237–248.

[4] R. B. Kellog, A. Tsan, *Analysis of some difference approximations for a singular perturbation problem without turning points*, Math. Comput. **32** (1978), 1025–1039.

[5] M. Stojanović, *An uniformly convergent quadratic spline difference scheme for singular perturbation problems*, (to appear in Mat. Ves.).

[6] K. Surla, M. Stojanović, *Singularly perturbed spline difference schemes on non-uniform grid*, (to appear in ZAMM).

[7] K. Surla, M. Stojanović, *Exponentially fitted spline difference schemes on non-uniform grid, in: Z. Bohte (ed)*, V Conf. Apl. Math., Proc. Symp., Ljubljana, 2–5 sept. 1986, 161–167.

Institut za matematiku
Dr Ilije Đuričića 4
21000 Novi Sad
Jugoslavija

(Received 12 11 1986)