PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 42 (56), 1987, pp. 149–153

ON A LIMIT THEOREM FOR RANDOM SEQUENCES

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Abstract. A limit theorem concerning sequences of maxima with random indices is proved under less restricted conditions than in [3].

Let $X_{n_1}, X_{n_2}, \dots, X_{nk_n}, \dots$ be a sequence of independent, identically distributed random variables for each $n = 1, 2, \dots$; $k_n \to \infty$ when $n \to \infty$, and let N_n be a sequence of non-negative random variables independent of X_{nk}.

Let us introduce the following notations:

$$P\{X_{nk} < x\} = F_n(x), \quad k = 1, 2, \dots$$
$$Y_{k_n} = \max_{1 \le k \le k_n} \{X_{nk}\}; \quad Y_{N_n} = \max_{1 \le k \le N_n} \{X_{nk}\}.$$

In [1], a version of "transfer,' theorem for maxima was proved, which gave sufficient conditions under which the weak convergence of distributions of maxima of a random number of random variables, follows from the convergence of distributions of maxima with nonrandom index. Namely, it was shown that from the conditions

(A) $\lim_{n\to\infty} P\{Y_{k_n} < x\} = F(x)$, and

(B) $\lim_{n\to\infty} P\left\{\frac{N_n}{k_n} < x\right\} = A(x)$, $A(0) = 0$,

it follows that the following condition (C) is fulfilled

(C) $\lim_{n\to\infty} P\{Y_{k_n} < x\} = G(x)$,

where

$$G(x) = \int_0^\infty ((F(x))^y \, dA(y).$$

A survey on the limit theorems for maxima with random indices could be found in [2].

AMS Subject Classification (1980): Primary 60F05.

The question whether (B) follows from the conditions (A) and (C) and (A) from (B) and (C), imposes itself naturally i w connection with the mentioned theorem. That was investigated in [3], where it was shown that the condition (A) follow, from (B) and (C). However, the proof that (A) and (C) imply (B), was given only in the following special (though important) case.

Let $X_1, X_2, \dots, X_n, \dots$ be a sequence of independent and identically distributed random variables, $Y_n = \max\{X_1, \dots, X_n\}$, such that for some special choice of normalizing constants $a_n > 0$ and b_n, the probability distribution of $a_n^{-1}(Y_n - b_n)$ converges, as $n \to \infty$, to a non-degenerate limit distribution $F(x)$. Namely, a special case of the initial problem, when $X_{nk} = a_n^{-1}(X_k - b)$ and $k_n = n$, was considered, and, accordingly, instead of the conditions (A), (B) and (C), the following conditions took their place:

(A$'$) $\lim\limits_{n\to\infty} P\{(Y_n - b_n)/a_n < x\} = F(x)$, and

(B$'$) $\lim\limits_{n\to\infty} P\{N_n/n < x\} = A(x)$, $A(0) = 0$,

(C$'$) $\lim\limits_{n\to\infty} P\{(Y_{N_n} - b_n)/a_n < x\} = G(x)$, where $G(x) = \int\limits_0^\infty ((F(x))^y \, dA(y)$.

It was shown that (A') and (C') imply (B'). It should be emphasized that in this case the class of all possible proper limiting probability distributions (determined by (A')), is the max-stable (or extremal) class which contains only three distribution types, while the class determined by the condition (A) is the class of all probability distributions. Here, we shall prove a version of this theorem under weaker conditions then (A') and (C'). An example, which shows that, if our conditions are not fulfilled, the statement of the theorem need not hold, will follow the proof of the theorem.

In the sequel we shall consider only proper probability distributions.

It is clear from the condition (C) that the sets of continuity points of F and G coincide.

By $\omega(F)$ we shall denote the upper endpoint of $F(x)$, defined by $\omega(F) = \sup\{x : F(x) < 1\}$.

Theorem. *Let the conditions (A) and (C) be fulfilled, let $F(x)$ be continuous at the upper endpoint and let us suppose that the sequence x_n, $n = 1, 2, \dots$ exists, such that*

$$\sum_i (-\log F(x_i))^{-1} = +\infty; \qquad 1 = F(x_1) > F(x_2) > \cdots > F(x_n) > \cdots \to 0.$$

Then the condition (B) is fulfilled.

Proof. We shall follow the same method of proof as in [3]. Namely, in order to prove that the sequence N_n/k_n converges, we shall prove that it converges completely, which is equivalent to stochastic boundedness of the sequence of corresponding probability distributions, and that the limit is unique.

Let us suppose that N_n/k_n is stochastically bounded. In that case, in order to prove (B), it suffices to prove that all convergent subsequences $P\{N_{n_r}/k_{n_r} < x\}$ of the sequence $P\{N_n/k_n < x\}$ have the same limit. If that were not true, then the subsequences n' and n'' of the sequence n would exist, such that

$$A_{n'}(x) = P\{N_{n'}/k_{n'} < x\} \underset{n'\to\infty}{\to} A_1(x), \quad \text{and}$$
$$A_{n''}(x) = P\{N_{n''}/k_{n''} < x\} \underset{n''\to\infty}{\to} A_2(x), \quad A_1(x) \neq A_2(x),$$

holds. But then, according to the transfer theorem, there would be

$$G(x) = \int_0^\infty ((F(x))^y \, dA_1(y) = \int_0^\infty ((F(x))^y \, dA_2(y), \quad x \in R \tag{1}$$

Let us denote by $L_1(t)$ and $L_2(t)$ the Laplace transforms of probability distributions $A_1(x)$ and $A_2(x)$, respectively. Then, we can rewrite (1) in terms of L_1 and L_2.

$$G(x) = L_1(-\log F(x)) = L_2(-\log F(x)), \quad x \in R. \tag{2}$$

The Laplace transforms L_1 and L_2 coincide on the set of points $\{s_n\}$, $s_n = -\log F(x_n)$, $n = 1, 2, \ldots$, which, by assumption satisfy the following conditions

$$\sum_i (s_i)^{-1} = +\infty, \text{ and } 0 = s_1 < s_2 < \cdots < s_n < \cdots \to +\infty \tag{3}$$

Laplace transform is uniquely determined by its values on the set of points satisfying the condition (3) (see [4], [5]), hence the corresponding probability distributions are uniquely determined too. Therefore, from (2) we deduce that $A_1 \equiv A_2$.

It remains to prove the stochastic boundedness of the sequence N_n/k_n. Suppose that this does not hold, i. e. that there is some $\varepsilon_0 > 0$, such that for every $r \in N$, there exists n, such that the following inequality is valid

$$P\left\{\frac{N_{n_r}}{k_{n_r}} \geq r\right\} \geq \varepsilon_0, \quad n_r \to \infty \text{ as } n \to \infty.$$

By the condition (A), we have

$$P\{Y_{k_{n_r}\cdot r} < x\} \underset{r\to\infty}{\to} F(x)$$

The sequence

$$P\{Y_{k_{n_r}\cdot r} < x\}, \quad r = 1, 2, \ldots$$

tends to zero as $r \to \infty$, for every $x < \omega(F)$. We have

$$P\{Y_{r\cdot k_{n_r}} < x\} = (F_{n_r}(x))^{r\cdot k_{n_r}},$$

and hence, there exists some $\varepsilon_1 > 0$, such that for each $x < \omega(F)$

$$\overline{\lim_{n\to\infty}}\, P\{Yr \cdot k_{n_r} > x\} \geq \varepsilon_1 > 0.$$

From the independence of N_n from X_{nk} from (3) and (4) and from the fact that, for $j > k$,

$$P\{Y_{nk} > x\} \le P\{Y_{nj} > x\},$$

it follows that for each $x < \omega(F)$, we have

$$\begin{aligned}\overline{\lim_{n\to\infty}} P\{Y_{N_n} > x\} &\ge \overline{\lim_{r\to\infty}} P\{Y_{N_{n_r}} > x\} \\ &\ge \overline{\lim_{r\to\infty}} \sum_{k=1}^{\infty} P\{\max_{1\le j\le k}\{X_{n_r j}\} > x\}\, P\{N_{n_r} = k\} \\ &\ge \overline{\lim_{r\to\infty}} \sum_{k\ge r\cdot k n_r} P\{\max_{1\le j\le k}\{X_{n_r j}\} > x\}\, P\{N_{n_r} = k\} \\ &= \varepsilon_1 \cdot P\{N_{n_r} \ge r\cdot k_{n_r}\} \\ &= \varepsilon_1 \cdot \varepsilon_0 > 0.\end{aligned}$$

When $x > \omega(F)$, we get that the limiting probability distribution G has a jump at the upper endpoint $\omega(F)$, contrary to the assumption of the theorem, which proves the stochastic boundedness of the sequence N_n/k_n. The proof is completed.

Example. Let us suppose that

$$\text{A)}\ \ F(x) = \lim_{n\to\infty} P\{Y_{k_n} < x\} = \begin{cases} 0 & x < 0 \\ x & 0 \le x < 1/2 \\ 1 & x \ge 1/2 \end{cases}$$

$$\text{C)}\ \ G(x) = \lim_{n\to\infty} P\{Y_{k_n} < x\} = \begin{cases} 0 & x < 0 \\ x/2 & 0 \le x < 1/2 \\ 1 & x \ge 1/2 \end{cases}$$

the continuity of F, G being violated at the upper endpoint $\omega(F) = \omega(G) = 1/2$. Then the corresponding N_n satisfies

$$P\left\{\frac{N_n}{k_n} < x\right\} = \begin{cases} 0 & x \le 1 \\ 1/2 & 1 < x \le n \\ 1 & x > n \end{cases}$$

and consequently

$$A(x) = \lim_{n\to\infty} P\left\{\frac{N_n}{k_n} < x\right\} = \begin{cases} 0 & x \le 1 \\ 1/2 & x > 1 \end{cases}$$

is an improper probability distribution.

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(Received 16 01 1987)