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FREDHOLM THEORY AND SEMILINEAR EQUATIONS WITHOUT RESONANCE INVOLVING NONCOMPACT PERTURBATIONS, II. APPLICATIONS

P. S. Milojević

1. Introduction. In this paper we shall give applications of the abstract theory developed in Part I [9] to nonlinear Hammerstein integral equations and to linear and nonlinear boundary value problems for semilinear elliptic equations. Due to the generality of the class of (pseudo) A-proper maps, we are able to treat nonlinear perturbations that depend also on the highest order derivatives, in contrast to the most of earlier known results. Moreover, our proofs are direct, i.e. do not require any reduction to an equivalent problem (using some type of inversion procedure), and are constructive (via finite-dimensional approximations) when the induced maps are A-proper. For applications to hyperbolic equations, we refer to [10].

2. Fredholm alternative to Hammerstein integral equations. Let $Q \subset R^n$ be a bounded domain, $F : Q \times R^n \to R^n$ and $K(t,s)$ be a $n \times n$ matrix, i.e. $K : Q \times Q \to R^{n^2}$. Let $L_2 = L_2(Q, R^n)$ and consider the Hammerstein integral equation

$$x(t) - \int_Q K(t,s)F(s,x(s))\,ds = h(t), \quad (t \in Q,\ h \in L_2) \tag{2.1}$$

Regarding F, we assume

(2.2) *$F(t,x)$ satisfies the Caratheodory condition, i.e. it is measurable in t for each fixed $x \in R^n$ and is continuous in x for each fixed $t \in Q$, and, for some $\lambda \in R$, $\beta > 0$ sufficiently small, $0 < p_k < 1$, $s_k \in L_{2/p_k}(Q,R)$ and $m \in L_2(Q,R)$,*

$$|F(s,x) - \lambda x| \le \beta x + \sum_{k=1}^{n} s_k(s)|x|^{1-p_k} + m(s).$$

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Define $C : L_2 \to L_2$ by $Cx(t) = \lambda \int_Q K(t,s)x(s)\,ds$ and assume that $K \in L_2(Q \times Q, R^{n^2})$. Since C is compact, $r = \dim N(I - C) < \infty$ and let $\{z_1, \ldots, z_r\} \subset L_2$ be a basis of the null space $N(I - C^*)$, i.e., they are linearly independent and

$$z_i(t) - \lambda \int_Q K(s,t) z_i(s)\,ds = 0 \quad (t \in Q,\ i = 1, \ldots, r). \tag{2.3}$$

Suppose also that for each $x \in L_2$

$$\int_Q (\int_Q K(t,s)[F(s,x(s)) - \lambda x(s)]\,ds) z_i(t)\,dt = 0, \quad 1 \le i \le r. \tag{2.4}$$

Set $Mx(s) = \lambda^{-1} CF(s, x(s))$ on L_2. Since $(M - C)x(t) = \int_Q K(t,s)[F(s,x(s)) - \lambda x(s)]\,ds$, the range $R(M - C) \subset N(I - C^*)^{\perp} = R(I - C)$ by (2.4). Moreover, $M : L_2 \to L_2$ is compact and Eq. (2.1) is equivalent to

$$x - Mx = h (h \in L_2). \tag{2.5}$$

Theorem 2.1 (Fredholm alternative). *Let* (2.2) hold. *Then*

(a) *If the equation $x - Cx = 0$ has a unique zero solution, Eq.* (2.1) *is approximation-solvable w.r.t. any Γ_0 for L_2 for each $f \in L_2$, or*

(b) *if $N(I - C) \neq \{0\}$ and* (2.4) *holds, then Eq.* (2.1) *is solvable for a given $h \in L_2$ if and only if*

$$\int_Q h(s) z_i(s)\,ds = 0, \quad (1 \le i \le r) \tag{2.6}$$

in which case there is a connected closed subset S of $(I - M)^{-1}(h)$ whose covering dimension at each point is at least r.

Proof. Set $A = I - C$, $N = M - C$ and $T = A - N$. Then $A : L_2 \to L_2$ is Fredholm of index zero with $\operatorname{codim} R(A) = r$. Moreover, T is A-proper w.r.t. any projection scheme $\Gamma_0 = \{X_n, P_n\}$ for L_2 and the quasinorm $|N|$ is sufficiently small (cf. [8]). Moreover, as in [8], we get that $R(N) \subset R(A) = N(A)^{*\perp}$. Hence, the conclusions follow from Theorems 2.3 and

Remark 2.1. Without the dimension assertion, Theorem 2.1 has been proved in [8] with i $|N| > 0$ and, existentially, in Kachurovsky [6] when $|N| = 0$. When $F(s, x(s)) = x(s)$, it reduces to the classical Fredholm alternative.

3. Nonlinear nonresonance perturbations of regular elliptic problems. In this section we shall consider the following semilinear elliptic boundary value problems without resonance (at zero or infinity)

$$Au - F(x, u, Du, \ldots, D^{2m}u) = f(x) \text{ in } Q \tag{3.1}$$

$$Au - G(x, u, Du, \ldots, D^{2m-1})u - F(x, u, Du, \ldots, D^{2m}u) = f(x) \text{ in } Q \tag{3.2}$$

(3.3) $B_j(x,D)u = 0$ on ∂Q, $j = 1, \dots, m$,

where the boundary ∂Q is smooth and $Au = \sum_{|\alpha|\leq 2m} a_\alpha(x)D^\alpha u(x)$ is an elliptic operator acting on $V = \{u \in W_p^{2m}(Q) \,|\, B_j u = 0 \text{ and } Q,\ 1 \leq j \leq m\}$, the space of functions satisfying "coercive" (i.e. Lopatinski-Schapiro) boundary conditions $B_j u = 0$ on ∂Q for some $p \in (1,\infty)$, with $a_\alpha \in C(\overline{Q})$ for $|\alpha| = |(\alpha_1, \dots, \alpha_n)| = \alpha_1 + \cdots + \alpha_n = 2m$ and $a_\alpha \in L_\infty(Q)$ for $|\alpha| < 2m$. Assume that $A : V \to L_p(Q)$ is Fredholm of index zero which is the case under suitable conditions on the boundary operators $\{B_j\}$. Let s_{2m} be the number of distinct derivatives of order $\leq 2m$.

Regarding F and G, we assume

(3.4) *$F : Q \times R^{s_{2m}} \to R$ satisfies the Caratheodory condition and there is $M > 0$ sufficiently small and $h \in L_p(Q)$ such that*

$$|F(x,\xi)| \leq h(x) + M \sum_{|\alpha|\leq 2m} |\xi_\alpha| \text{ for a.e. } x \in Q \text{ and each } \xi \in R^{s^2 m}.$$

(3.5) *$G : Q \times R^{s_{2m-1}} \to R$ satisfies (3.4) on $Q \times R^{s_{2m-1}}$ and, for each $u \in W_p^{2m-1}(Q)$, $B(u) = G(x,u,Du,\dots,D^{2m-1}u)$ is a continuous linear map from L_p into itself and such that*

$$\limsup_{||u||_{2m\to\infty}} ||B(u)|| < \frac{1}{||A^{-1}||_{L_p \to L_p}}$$

Define: $N : V \to L_p$ by $Nu = F(x,u,\dots D^{2m}u)$. Then boundary value problems (3.1), (3.3) and (3.2), (3.3) are equivalent to the operator equations

$$Au - Nu = f \tag{3.6}$$

and

$$Au - B(u)u - Nu = f. \tag{3.7}$$

When A is injective, we have

Theorem 3.1. *$A : V \to L_p$ be injective and A-proper w.r.t. $\Gamma = \{X_n, Y_n, Q_n\}$ for (V, L_p) and (3.4)–(3.5) hold. Then*

(a) *If $A - N : V \to L_p$ is (pseudo) A-proper w.r.t. Γ, BVP (3.1), (3.3) is (solvable) approximation-solvable for each $f \in L_p$.*

(b) *If $N : V \to L_p$ is k-ball contractive, $k < 1$, then BVP (3.2), (3.3) is approximation-solvable for each $f \in L_p$.*

Proof. (a) By (3.4), $||Nu|| \leq a + b||u||$ for each $u \in V$ and some a and b. Since b is sufficiently small, the conclusions follow from Theorem 2.1 in [9]. (b) Define a map $U : V \times V \to L_p$ by $U(u,v) = B(u)v$. For each $v \in V$ fixed, the map $U(\cdot, v) = B(\cdot)\,v : V \to L_p$ is completely continuous by the continuity of $u \to G(x,u,\dots,D^{2m-1})$ from W_p^{2m-1} into L_p and the compactness of the embedding $V \hookrightarrow W_p^{2m-1}$. Moreover, for each $u \in V$ fixed, the map $U(u,\cdot) = B(u)(\cdot) : V \to L_p$

is also completely continuous. Hence, $T_1 u = Au - B(u)u$ and $T = T_1 - N$ are A-proper w.r.t. Γ by Proposition 3.1 in [9]. Thus, the conclusion follows from Theorem 3.3 in [9] . □

Let us now look at some special cases. Suppose

(3.8) *There is a constant k_1 sufficiently small such that*

$$|F(x,\eta,\xi) - F(x,\eta,\xi')| \le \sum_{|\alpha|\le 2m} |\xi_\alpha - \xi'_\alpha|$$

for a.e. $x \in Q$ and all $\eta \in R^{s_{2m-1}}$, $\xi, \xi' \in R^{s'_{2m}}$, $s'_{2m} = s_{2m} - s_{2m-1}$.

Corollary 3.1. *Let A be as in Theorem* 3.1, *and* (3.4), (3.5) *and* (3.8) *hold. Then BVP's* (3.1), (3.3) *and* (3.2), (3.3) *are approximation-solvable for each $f \in L_p$.*

Proof. Define a map $U : V \times V \to L_p$ by

$$U(u,v) = F(x,u,\ldots,D^{2m-1}u, D^{2m}v).$$

Then, for each $v \in V$ fixed, $U(\cdot,v) : V \to L_p$ is continuous, bounded and therefore compact by the imbedding theorem. Moreover, for each $u \in V$ fixed, $U(u,\cdot) : V \to L_p$ is k-ball-contractive with k small. Since $Nu = U(u,u)$ for $u \in V$, it is k-ball-contractive and therefore $A - N$ is A-proper w.r.t. Γ. Hence, Corollary 3.1 follows from Theorem 3.1. □

Corollary 3.2. *Let A be as in Theorem* 3.1 *and $F(x,u,\ldots,D^{2m}u) = F_1(x,u,\ldots,D^{2m-1}u) + F_2(x,Au)$ such that*

(3.9) *$N_1 u = F_1(x,u,\ldots,D^{2m-1}u)$ is continuous form W_p^{2m-1} into L_p;*

(3.10) *$F2 : Q \times R \to R$ is continuous and there are $h \in L_p$ and $b > 0$ such that for $x \in Q$, $t, t_1, t_2 \in R$:*

$$|F_2(x,t)| \le b(|h(x)| + |t|) \text{ and } (F_2(x,t_1) - F_2(x,t_2))(t_1 - t_2) \ge 0.$$

Then BVP (3.1), (3.3) *is solvable for each $f \in L_p$.*

Proof. Let $N_2 u = F_2(x,Au)$ on V. Then $A - N_1 + N_2 : V \to L_p$ is pseudo A-proper w.r.t. $\Gamma = \{X_n, A(X_n), Q_n\}$ since N_1 is completely continuous and N_2 is A-monotone, i.e. $(N_2 u - N_2 v, A(u-v)) \ge 0$ on V (cf. [12]). Since $||(N_1 + N_2)u|| \le a + b||u||_{2m}$ for $u \in V$ with b sufficiently small, the corollary follows from Theorem 3.1 (a). □

Let us now look at (3.1)–(3.3) when $\ker A \ne \{0\}$ and there is no resonance at infinity. Suppose that $F = F_1 + F_2$ satisfies the Carathéodory condition and

(3.11) *There are $h_1 \in L_p$, $M_1 > 0$ and $\delta \in (0,1)$ such that*

$$|F_1(x,\xi)| \le h_1(x) + M_1 \sum_{|\alpha|\le 2m} |\xi_\zeta|^\delta \text{ for a.e. } x \in Q, \text{ all } \xi \in R^{s_{2m}}$$

(3.12) *There are* $\lambda \in R$, $b_\alpha \geq 0$ *and* $h_2 \in L_p$ *such that for a.e* $x \in Q$ *and* $\xi \in R^{s_{2m}}$

$$|F_2(x,\xi_0,\xi_1,\dots\xi_{2m}) - \lambda\xi_0| \leq h_2(x) + \sum_{|\alpha|\leq 2m} b_\alpha|\xi_\alpha|.$$

THEOREM 3.2. *Let* $A_\lambda = A - \lambda I : V \to L_p$ *have a continuous inverse and* $\Gamma = \{X_n = A_\lambda^{-1}(Y_n), Y_n, Q_n\}$ *be a scheme for* (V, L_p). *Then*

(a) *If* (3.11)–(3.12) *hold with the* b_α*'s small and* $A - N : V \to L_p$ *is (pseudo) A-proper w.r.t.* Γ, *then BVP* (3.1), (3.3) *is (solvable) approximation-solvable for each* $f \in L_p$.

(b) *If* (3.4) *hold* $N : V \to L_p$ *is k-ball-contractive with* $k < 1$ *and* $B_\lambda = B - \lambda I$ *and* A_λ *satisfy* (3.5), *then BVP* (3.2), (3.3) *is approximation-solvable for each* $f \in L_p$.

Proof. (a) We note first that $||Q_nA_\lambda u|| = ||A_\lambda u|| \geq ||A_\lambda^{-1}||\, ||u||_{2m}$ for $u \in X_n$. Let $N_iu = F_i(x,u,\dots D^{2m}u)$, $i = 1,2$, and $N = N_1 + N_2$. By (3.12), Minkowski and Hölder inequalities imply that

$$||N_2u - \lambda u|| \leq a + b||u||_{2m} \text{ for } u \in V$$

and some a and b. Moreover, $N_1 : V \to L_p$ has a sublinear growth by (3.11), and therefore $||N - \lambda I|| < b$. Hence, Theorem 2.1 [9] applies.

(b) Using the arguments similar to those in the proof of Theorem 3.1 (b), we see that the conclusion follows from Theorem 3.3 [9]. □

Next, suppose that $A : D(A) = V \subset L_2 \to L_2$ is self-adjoint and has the pure point spectrum consisting of eigenvalues: $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \leq \dots$ of finite, multiplicities, with no finite point of accumulation. Assume

(3.13) *Let* $\lambda_k < \lambda_{k+1}$ *and* λ *and* γ *be such that* $\lambda_k < \lambda < \lambda_{k+1}$ *and* $0 \leq \gamma < \min\{\lambda - \lambda_k, \lambda_{k+1} - \lambda\}$ *and small. Suppose that for some* $h_2 \subset L_2$, *a.e.* $x \in Q$ *and all* $\xi = (\xi_0,\xi_1,\dots,\xi_{2m}) \in R^{s_{2m}}$

$$|F_2(x,\xi) - \lambda\xi_0| \leq \gamma|\xi_0| + h_2(x).$$

(3.14) *There are positive constants* ε, ϱ *and* M *and* $h_1 \in L_2$ *such that*
(i) $|F_2(x,s,\xi)| \leq M|s| + h_1(x)$ *for a.e.* $x \in Q$, $s \in R$, $\xi \in R^{s_{2m}-1}$;
(ii) *For some* $\lambda_k < \lambda_{k+1}$ *and all* $(x,s,\xi) \in Q \times R \times R^{s_{2m}-1}$ *with* $|s| \geq \varrho$

$$\lambda_k + \varepsilon \leq F_2(x,s,\xi)/s \leq \lambda_{k+1} - \varepsilon.$$

Note that (ii) holds if $\lim_{s\to\pm\infty} F_2(x,s,\xi)/s = f_\pm(x)$ uniformly with respect to $\xi \in R^{s_{2m}-1}$ and $f_\pm(x) \in [\lambda_k + \varepsilon, \lambda_{k+1} - \varepsilon]$ for $x \in Q$. Moreover, (3.14) implies (3.13) with $\lambda = (\lambda_k + \lambda_{k+1})/2$, $\gamma = (\lambda_{k+1} - \lambda_k)/2$ and $h_2(x) = h_1(x) + \varrho(M + \lambda)$. We have now

THEOREM 3.3. *Let* $A : V = W_2^{2m}(Q;\{B_j\}) \to L_2$ *be Fredholm of index zero and self-adjoint in* L_2. *Then*

(a) *If* (3.11) *and* (3.13) *hold and* $A-N : V \to L_2$ *is* (*pseudo*) A*-proper w.r.t.* $\Gamma = \{X_n = A^{-1}(Y_n), Y_n, Q_n\}$, *then BVP* (3.1), (3.3) *is* (*solvable*) *approximation-solvable for each* $f \in L_2$.

(b) *If* (3.11) *and* (3.13) *hold,* $N : V \to L_2$ *is k-ball-contractive,* (*e.g.* (3.8) *holds*), $k < 1$, *and*

$$\limsup_{||u||_{2m}\to\infty} ||B(u) - \frac{1}{2}(\lambda_k + \lambda_{k+1})I|| < \frac{1}{2}(\lambda_{k+1} - \lambda_k), \tag{3.15}$$

then BVP (3.2), (3.3) *is approximation-solvable for each* $f \in L_2$.

Proof. (a) The spectral gap of A induced by the gap $(\lambda_k, \lambda_{k+1})$ is $(\lambda_k - \lambda, \lambda_{k+1} - \lambda)$. Hence, $A_\lambda^{-1} : L_2 \to L_2$ is a bounded self-adfoint map whose spectrum lies in $[(\lambda_k - \lambda)^{-1}, (\lambda_{k+1} - \lambda)^{-1}]$ and so $||A_\lambda^{-1}|| \le \max\{(\lambda - \lambda_k)^{-1}, (\lambda_{k+1} - \lambda)^{-1}\} \le 1/\gamma$. Since A is self-adjoint in L_2, it follows that $0 < \gamma < ||A_\lambda^{-1}||^{-1} = \min\{|\mu| \mid \mu \in \sigma(A - \lambda I)\}$.

Next, let $N_i u = F(x, u, \ldots, D^{2m}u)$ on V, $i = 1, 2$. Using Minkowski and Hölder inequalities we get $||N_2 u - \lambda u|| \le \gamma||u|| + ||h_2||$ for $u \in V$ and $||N_1 u|| \le m||u||_{2m}^{\delta} + ||h_1||$ for some m and all $u \in V$. Hence, $||Nu - \lambda u|| \le \gamma||u|| + m||u||_{2m}^{\delta} + b$ for $u \in V$ and the conclusions follow from Theorem 3.1 in [9]. (b) Let $\lambda = (\lambda_k + \lambda_{k+1})/2$. Then, as in (a), $||A_\lambda^{-1}|| = (\lambda_{k+1} - \lambda_k)/2$. Hence, as in Theorem 3.1, the conclusion follows from Theorem 3.3 in [9]. □

As before, we obtain

Corollary 3.2. *Let A be as in Theorem* 3.2 *and* (3.11) *and* (3.13) *hold. Then, if* $F = F_1 + F2$ *satisfies* (3.8) ((3.9)–(3.10), *respectively*), *Theorem* 3.2 (a) *holds.*

Remark 3.1. Clearly, (3.8) holds if F does not depend on $D^\alpha u$ with $|\alpha| = 2m$. In this setting, Corollary 3.2 was proved existentially by de Figueiredo [4], while Theorem 3.3 (b) by Kazdan-Warner [7], using completely different arguments. When $F = F_2$ satisfies (3.8) and (3.14) with $m = 1$ and A is symmetric and uniformly elliptic in $\overline{Q}$, Corollary 3.2 also extends one of the main existence results of Fitzpatrick [5]. His proof was based on the Courant min-max principle and therefore does not extend to the higher order equations. As in [7], we could extend Theorem 3.2 to systems of equations in (3.1) and (3.2).

4. Nonlinear boundary value problems for semilinear elliptic equations. Let Q be a bounded open subset of R^n with smooth boundary and $\beta : R^1 \to 2^{R^1}$ have a maximal monotone graph in R^2 with $0 \in \beta(0)$. Let $X = W_2^2(Q)$, $Y = L_2(Q)$ In this section we shall study the solvability of the following eigenvalue problems in X.

$$\begin{cases} -\Delta u + F(x, u, \nabla u, \Delta u) = \lambda u + f \text{ in } Q \\ -\partial u/\partial n \in \beta(u) \text{ on } \partial Q. \end{cases} \tag{4.1}$$

and

$$\begin{cases} -\Delta u + \gamma(u) \ni \lambda u + f \text{ in } Q \\ -\partial u/\partial n \in p(u) \text{ on } \partial Q, \end{cases} \tag{4.2}$$

where $f \in L_2(Q)$ and $\partial/\partial n$ denotes the outward normal derivative, and γ is another maximal monotone graph in R^2. Such problems appear in thermodynamics, fluid dynamics, elasticity, etc.

4.1. Let us first consider the eigenvalue problem (4.1). Let $C = I : X \to Y$ be the natural imbedding which is compact by the imbedding theorem. We make the following assumptions on the nonlinear term

(4.3) *$F : Q \times R^{s_2} \to R^1$ satisfies the Caratheodory condition, $F = F_1 + F_2$ and*

(i) *there are positive constants b_α, $\delta \in (0,1)$ and $h_1 \in L_2$ such that $|F_1(x,\xi)| \leq h_1(x) + \sum_{|\alpha|\leq 2} b_\alpha |\xi_\alpha|^\delta$ for $x \in Q$ (a.e.) and $\xi \in R^{s_2}$.*

(ii) *There are $\lambda_1 \in R^+$, $c_\alpha > 0$ sufficiently small and $h_2 \in L_2$ such that $|F_2(x,s,\xi) - \lambda_1 s| \leq h_2(x) + \gamma \sum_{|\alpha|\leq 2} c_\alpha |\xi_\alpha|$ for $x \in Q$ (a.e.) and $\xi \in R^{s_2-1}$, $s \in R^1$.*

(4.4) $F(x,\eta,\xi)$ is continuous in (x,η) uniformly with respect to $\xi \in R^1$ and there is a constant $M > 0$ such that

$$|F(x,\eta,\xi_1) - F(x,\eta,\xi_2)| \leq M|\xi_1 - \xi_2| \text{ for } x \in Q \ (a.e), \eta \in R^n, \ \xi_1, \xi_2 \in R^1.$$

Let $D(A) = \{u \in W_2^2(Q) \mid -\partial u/\partial n \in \beta(u) \text{ a.e. on } \partial Q\}$ and define $Au = -\Delta u$ for $u \in D(A)$. We note that $D(A)$ is well defined by the trace theorem and $A : D(A) \subset Y \to Y$ is a maximal monotone (nonlinear) mapping. It is well known that ([1]) $C(A + \lambda_1 I)^{-1}$ is nonexpansive and compact in Y and there is a constant K such that

$$||u||_X \leq K(||Au + \lambda_1 u||_Y + 1) \text{ for } x \in D(A). \tag{4.5}$$

THEOREM 4.1. *Suppose that* (4.3) *and* (4.4) *hold. Then there exists an $\lambda_0 > 0$ such that for each $\lambda \in (-\lambda_0, \lambda_0)$ BVP* (4.1) *has a solution for each f in L_2.*

Proof. Define $N_i : X \to Y$ by $N_i u = F_i(x, u, \nabla u, \Delta u)$, $i = 1, 2$. Then there are constants m_i depending only on the b_α's and c_α's such that for each $u \in X$

$$||N_1 u|| \leq m_1 ||u||^\delta + ||h_1|| \text{ and } ||N_2 u - \lambda_1 u|| \leq \gamma m_2 ||u|| + ||h_2||.$$

Since γm_2 is suitably small, there exists an $R > 0$ such that $Ka < 1$ for $a = \gamma m + m_1 R^{\delta - 1}$ and therefore $N = N_1 + N_2$ satisfies

$$||Nu - \gamma_1 u|| \leq a||u|| + b \text{ for } ||u|| \geq R, \text{ where } b = ||h_1|| + ||h_2||.$$

Let $\lambda_0 > 0$ be such that $K(a + \lambda_0) \leq 1$. Then, $||Nu - (\lambda_1 + \lambda)u|| \leq (a + |\lambda|)||u|| + b$ for each $\lambda \in (-\lambda_0, \lambda_0)$ and $||u|| \geq R$. Therefore, since $N(A + (\lambda_1 + \lambda)I)^{-1}$ is M-ball-contractive in Y as in [5], the conclusion follows from Theorem 3.5 [9]. □

4.2. In this section we shall first consider a class of general eigenvalue problems in a Banach space X and then we shall obtain some solvability results for the nonlinear eigenvalue problem (4.2).

Let $T : D(T) \subset X \to 2^X$ be an m-accretive mapping (i.e., $\lambda I + T$ is surjective for each $\lambda > 0$ and T is accretive) and consider the eigenvalue problem

$$f + \lambda x \in Tx \quad (f \in X \text{ given}). \tag{4.6}$$

By the m-accretivity of T, (4.6) is uniquely solvable for each f if $\lambda < 0$. Suppose that $\lambda \geq 0$ and $\lambda_0 > 0$ is fixed. Setting $\mu = \lambda + \lambda_0$, $y = \mu x + f$ and $N = N_{\lambda_0} = (\lambda_0 I + T)^{-1} : X \to X$, it is easy to see that (4.6) is equivalent to

$$y - \mu N y = f. \tag{4.7}$$

Since N is nonexpansive, μN is a μ-contractive mapping and therefore (4.7) is uniquely solvable for each f if $\mu < 1$. Hence, it remains to consider the solvability of (4.6) when $\lambda \geq 1$. In view of the above discussion, as an immediate consequence of Theorem 2.1 (c) in [9] we obtain

THEOREM 4.2. *Let $I - (\lambda + \lambda_0)N : X \to X$ be pseudo A-proper w.r.t. a scheme $\Gamma = \{X_n, P_n\}$ for some $\lambda \geq 1$ and $A : X \to X$ be such that for some $c_0 > 0$ and $n_0 \geq 1$,*

$$||x - (\lambda + \lambda_0)P_n Ax|| \geq c_0||x|| \text{ for } x \in X_n,\ n \geq n_0. \tag{4.8}$$

Suppose that

$$|N - A| = \limsup_{||x|| \to \infty} \frac{||Nx - Ax||}{||x||} < \frac{1}{\lambda c}.$$

Then (4.6) *is solvable for each f in X.*

It follows that if, for example, A and N are compact with A linear and $1/\lambda$ is not an eigenvalue of A, then all the hypotheses of Theorem 4.2, except (4.9), are satisfied. Hence, if (4.9) also holds, then (4.6) is solvable. We shall see below that this situation occurs in studying (4.2).

Assume that β and $\gamma : R^1 \to 2^{R^1}$ have maximal monotone graphs in R^2 such that $0 \in \beta(0) \cap \gamma(0)$ and the domains $D(\beta)(= \{y \in R^1 \mid \beta(y) \neq \emptyset\})$ and $D(\gamma)$ contain at least one half-line starting at the origin. The classes of monotone graphs to be considered in (4.2) are defined next.

Definition 4.1. Let $\alpha : R^1 \to 2^{R^1}$ be such that $D(\alpha)$ contains at least one half-line starting at the origin.

(a) α is said to be *asymptotically close* to $\alpha(\infty) = \alpha' < \infty$ if there is a constant $a \geq 0$ such that to each $\varepsilon > 0$ there corresponds an R_ε such that

$$|z - \alpha' y| \leq (a + \varepsilon)|y| \text{ for each } z \in \alpha(y),\ y \in D(\alpha),\ |y| \geq R_\varepsilon$$

and we write $|\alpha - \alpha'| = a$.

(b) α is said to satisfy *condition* (+) if to each $M > 0$ there corresponds an $R_M > 0$ such that

$$|z| \geq M|y| \text{ for each } z \in \alpha(y),\ y \in D(\alpha) \text{ and } |y| \geq R_M.$$

Define $B(U) = -\Delta u$ for $u \in D(B) = \{u \in W_2^2(Q) \mid -\partial u/\partial n \in \beta(u(x)) \text{ for } x \in \partial Q \text{ a.e.}\}$. Then B is a maximal monotone operator in $H = L_2(Q)$ and, for each $\lambda > 0$ fixed, there is a constant $c > 0$ such that

$$||u||_{2,2} \leq c|| - \Delta + \lambda u||_2 \text{ for each } u \in D(B) \tag{4.10}$$

by Theorem 10 in [1]. Moreover, by Corollary 13 in [2], $Tu = -\Delta u + \gamma(u)$ with $u \in \{u \in W_2^2(Q) \mid -\partial u/\partial n \in \beta(u(x)),\ u(x) \in D(\gamma) \text{ for } x \in \partial Q \text{ a.e.}\}$ is maximal monotone in H and therefore, for each $u \in H$ there is a unique function Nu satisfying

$$-\Delta Nu + \gamma(Nu) + \lambda_0 Nu \ni u \text{ in } Q \tag{4.11}$$

$$-\partial Nu/\partial n \in \beta(Nu) \text{ on } \partial Q \tag{4.12}$$

for any $\lambda_0 > 0$ fixed. Moreover, $N = N_{\lambda_0} = (\lambda_0 I + T)^{-1} : H \to H$ is nonexpansive and $||Nu|| \leq ||u||$ for each $u \in H$.

Let us now consider (4.2). It is easy to see that it possesses only the trivial solution $u = 0$ if $\lambda < 0$. Suppose that $\lambda \geq 0$ from now on. Then we can write (4.2) in the operator form as

$$f + \lambda u \in Tu,\ u \in D(T) \tag{4.13}$$

which is equivalent to (cf. [3])

$$v - \mu N v = f, \quad v \in L_2, \tag{4.14}$$

where $\mu = \lambda_0 + \lambda$, $v = \mu u + f$ and N is defined by (4.11)–(4.12).

In order to apply Theorem 4.2 we need to find a mapping A asymptotically close to N. To that end, the following result is needed.

Proposition 4.1. (a) *Let $\alpha : R^1 \to 2^{R^1}$ satisfy condition $(+)$ and $G \subset R^n$ be measurable. If $u, v \in L_2(G)$, $u(x) \in D(\alpha)$ and $v(x) \in \alpha(u(x))$ for $x \in G$ (a.e.), then there is a $c > 0$ such that to each $M > 0$ there corresponds $R_M > 0$ such that $||u||_2 \leq c(R_M + M^{-1}||v||_2)$.*
(b) *Let $\alpha : R^1 \to 2^{R^1}$ be asymptotically close to $\alpha(\infty) = \alpha'$ and bounded (i.e. maps bounded sets into bounded sets). If $u, v \in L_2(G)$, $u(x) \in D(\alpha)$ and $v(x) \in \alpha(u(x))$ for $x \in G$ (a.e.), then there exists a $c > 0$ such that to each $\varepsilon > 0$ there corresponds a $C_\varepsilon > 0$ such that*

$$||v - \alpha' u||_2 < c((a + \varepsilon)\,||u||_2 + C_\varepsilon).$$

Proof. Part (a) was proved in [3] and (b) is an extension of Lemma 1.2 in [3]. Setting $A = \{x \mid u(x) > R_\varepsilon\}$, its conclusion follows from

$$||v - \alpha' u||_2^2 = \int_A (v(x) - \alpha' u(x))^2\, dx + \int_{B = G - A} (v(x) - \alpha' u(x))^2\, dx$$

$$\leq \int_A (a + \varepsilon)^2 u^2(x)\, dx + \int_B (v(x) - \alpha' u(x))^2\, dx \leq c^2((a + \varepsilon)\,||u||_2^2 + C_\varepsilon^2),$$

where we used at the last step that $|u(X)| \leq R_\varepsilon$ and α is bounded. □

The following Green's theorem will be used in the sequel: if $u(x) \leq W_2^2(Q)$ and $w \in W_q^1(Q)$ then

$$-\int_Q \Delta \cdot w dx = \sum_i \int_Q \frac{\partial u}{\partial x_i} \frac{\partial w}{\partial x_i} dx - \int_Q \frac{\partial u}{\partial n} w d\sigma$$

where $1 < q \leq 2$ if $n = 2$ and $q = 2n/(n+2)$ if $n > 2$.

Proposition 4.2. *Let γ be asymptotically close to γ' and bounded. Then* $||Nu||_{2,1} \leq \sqrt{2-\lambda_0}\,||u||$ *and* $||Nu||_{2,2} \leq c((1+a+\varepsilon)\,||u|| + C_\varepsilon)$ *for* $u \in H$, *and some* $c > 0$ *with* $\varepsilon > 0$ *given.*

Proof. (Cf. also [3].) *Let*$u \in H$ be fixed. Then there exists an $w \in \gamma(Nu)$ such that

$$-\delta Nu + w + \lambda_0 Nu = u \text{ in } Q. \tag{4.15}$$

Since β and γ are monotone and $0 \in (\beta(0) \cap \gamma(0))$, we have that

$$(-\partial Nu/\partial n, Nu)_{L_2(\partial Q)} \geq 0$$

and $(w, Nu)_H \geq 0$. Multiplying (4.15) by Nu and using the Green theorem it follows that

$$\lambda_0 ||Nu||^2 = \int\limits_Q (\Delta Nu + u - w) Nu\,dx = -\int\limits_Q |\nabla Nu|^2\,dx + \int\limits_{\partial Q} \frac{\partial}{\partial n}(Nu)\,Nu\,d\sigma +$$
$$+ \int\limits_Q (u-w) Nu\,dx \leq -\int\limits_Q |\nabla Nu|^2\,dx + \int\limits_Q uNu\,dx.$$

Hence, $\lambda_0 ||Nu||^2 + ||\nabla Nu||^2 \leq ||u||\,||Nu|| \leq ||u||^2$ or,

$$||Nu||_{2,1}^2 \leq (1-\lambda_0)\,||Nu||^2 + ||u||^2 \leq (2-\lambda_0)\,||u||^2,$$

so that $||Nu||_{2,1} \leq (2-\lambda_0)^{1/2} ||u||$ for each $u \in H$.

Next, it follows from (4.10) and Proposition 4.1 that for a given $\varepsilon > 0$

$$||Nu||_{2,2} \leq c||u-w|| \leq c(||u|| + (a+\varepsilon)\,||Nu|| + C_\varepsilon) \leq c((1+a+\varepsilon)\,||u|| + C_\varepsilon). \;\Box$$

Next we shall prove the existence of a linear mapping A to which N is asymptotically close.

Proposition 4.3. *Suppose that satisfies condition* $(+)$ *and* γ *is bounded and asymptotically close to* $\gamma(\infty) = \gamma'$. *Let* N *be defined by* (4.11)–(4.12) *and a linear mapping* $A : H \to H$ *be given by*

$$\begin{cases} \Delta Au + \gamma' Au + \lambda_0 Au = u \text{ in } Q \\ Au = 0 \text{ on } \partial Q. \end{cases}$$

Then the conclusions of Proposition 4.2 *hold for* A *and for each* $\varepsilon > 0$ *there exists* $R_\varepsilon > 0$ *such that* $||Nu - Au||_{2,1} \leq c(a+\varepsilon)\,||u||$ *for each* $||u|| \geq R_\varepsilon$.

Proof. (See also [3].) Since A is defined involving maximal monotone graphs that are special cases of those ones defining N, the first assertion of the proposition is valid.

Next, let $\varepsilon > 0$ be given and $u \in H$ fixed. There exists an $w \in \gamma(Nu)$ such that

$$-\Delta Nu + w + \lambda_0 Nu = u \text{ in } Q;\; -\partial Nu/\partial n \in \beta(Nu) \text{ on } \partial Q.$$

Therefore, $Nu - Au = \Delta(Nu - Au) + \gamma' Au - w$, and, after multiplying this equality by $Nu - Au$ and using the Green theorem, we obtain that

$$||Nu - Au||^2 = -\int_Q |\nabla(Nu - Au)|^2 \, dx + \int_{\partial Q} \frac{\partial}{\partial Q}(Nu - Au)Nu \, d\sigma + \\ + \int_Q (\gamma' Au - w)(Nu - Au) \, dx$$

since $Au = 0$ on ∂Q. Since $0 \in \beta(0)$, β is monotone and $-\partial Nu/\partial n \in \beta(Nu)$, $(-\partial Nu(x)/\partial n, Nu(x)) \geq 0$ on ∂Q and therefore,

$$\begin{aligned} ||Nu - Au||^2_{2,1} &= \int_{\partial Q} \frac{\partial}{\partial n}(NU - Au)\, Nu \, d\sigma + \int_Q (\gamma' Au - w)(Nu - Au) \, dx \\ &\leq -\int_{\partial Q} \frac{\partial Au}{\partial n} Nu \, d\sigma - \gamma' ||Nu - Au||^2 + \int_Q (\gamma' Au - w)(Nu - Au) \, dx \\ &\leq ||\partial Au/\partial n||_{L_2(\partial Q)} \, ||Nu||_{L_2(\partial Q)} + ||\gamma' Nu - w|| \, ||Nu - Au||. \end{aligned}$$

Moreover, $||\partial Au/\partial n||_{L_2(\partial Q)} \leq c||u||$, $||Nu - Au|| \leq ||Nu - Au||_{2,1} \leq c||u||$ and, by Proposition 4.1 (a)–(b) for β and γ respectively, for $M > 0$ and $\varepsilon > 0$ there are R_M and C_ε such that

$$||Nu||_{L_2(\partial Q)} \leq c(R_M + M^{-1}||\partial Nu/\partial n||_{L_2(\partial Q)}, \; ||\gamma' Nu - w|| \leq c((a + \varepsilon)||u|| + C_\varepsilon).$$

Since $||Nu||_{2,2} \leq c((1 + a + \varepsilon)\,||u|| + C_\varepsilon)$, it follows that

$$||Nu - Au||^2_{2,1} \leq c(R_M + M^{-1}(1 + a + \varepsilon)\,||u|| + c(a + \varepsilon)\,||u||C_\varepsilon M^{-1} + 1).$$

Hence, for a fixed $\varepsilon > 0$ we can choose M and R_ε large enough so that

$$||Nu - Au||^2_{2,1} \leq c^2(a + \varepsilon)\,||u||^2 \text{ for } ||u|| > R_\epsilon. \; \square$$

Our first result now for (4.2) is

THEOREM 4.3. *Suppose that β satisfies condition (+) and γ is bounded and asymptotically close to $\gamma(\infty) = \gamma'$ with $|\gamma - \gamma'| = a$ sufficiently small. If $\lambda \geq 1$ is not an eigenvalue of*

$$\begin{cases} -\Delta u + \gamma' u = \lambda u \text{ in } Q \\ u = 0 \text{ on } \partial Q, \end{cases} \tag{4.17}$$

then (4.2) is solvable in $W^2_2(Q)$ for each f in H.

Proof. Let N be defined by (4.11)–(4.12) with $\lambda_0 > 0$ small and $\mu = \lambda + \lambda_0$. By our discussion above it suffices to solve (4.14) in H for each f and this will be done using Theorem 4.2. The second inequalities in Propositions 4.2 and 4.3 imply that A, $N : H \to H$ are compact and continuous, respectively. Hence, $I - \mu N$ and

$I-\mu A$ are A-proper w.r.t. a projection scheme $\Gamma = \{X_n, P_n\}$ for H. Moreover, the null space of $I-\mu A$ is trivial since λ is not an eigenvalue of (4.17) and consequently (4.8) holds. Since a is sufficiently small, we see that (4.9) holds by Proposition 4.3 with $\varepsilon = \lambda_0$ and therefore (4.14) is solvable for each f by Theorem 4.2. $\square$

A similar result holds if condition (+) is replaced by the asymptotic one. We have

Theorem 4.4. Suppose that β and γ are bounded and asymptotically close to β' and γ' respectively with $\beta-\beta'|$ and $|\gamma-\gamma'|$ sufficiently small. If $\lambda \geq 1$ is not an eigenvalue of

$$\begin{cases} -\Delta u + \gamma' u = \lambda u \text{ in } Q \\ -\dfrac{\partial u}{\partial n} = \beta' u \text{ on } Q \end{cases}$$

then (4.2) is solvable in $W_2^2(Q)$ for each f in H.

The proof of Theorem 4.4 is similar to that of Theorem 4.3 and is based on the following result.

PROPOSITION 4.4. *Suppose that β and y are bounded and asymptotically close to β' and y' respectively. Let N be defined by* (4.11)–(4.12) *and a linear mapping $A: H \to H$ be given by*

$$-\Delta Au + \gamma' Au + \lambda_0 Au = u \textit{ in } Q$$
$$-\partial Au/\partial n = \beta' Au \textit{ on } Q.$$

Then Proposition 4.2 *is valid for A and for each $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that $||Nu - Au||_{2,1} \leq c(|\beta-\beta'| + |\gamma-\gamma'| + 3\varepsilon)||u||$ for each $||u|| \geq R_\varepsilon$.*

Proof. As in Proposition 4.3, the conclusions of Proposition 4.2 are valid for A. Since $\partial/\partial n(Nu - Au) = \partial Nu/\partial n + \beta' Nu$, as in the proof the Proposition 4.3 we obtain for each $u \in H$ that

$$\begin{aligned} ||Nu - Au||_{2,1}^2 &= \int_{\partial Q} \frac{\partial}{\partial n}(Nu - Au)\,(Nu - Au)\,d\sigma + \int_Q (\gamma' Au - w)\,(Nu - Au)\,dx \\ &= -\beta' ||Nu - Au||_{L_2(\partial Q)} - \gamma' ||Nu - Au||^2 + \int_{\partial Q} \left(\frac{\partial Nu}{\partial n} + \beta Nu\right)\,(Nu - Au)\,d\sigma + \\ &+ \int_Q (\gamma' Nu - w)\,(Nu - Au)\,dx \leq \left|\left|\frac{\partial Nu}{\partial n} + \beta' Nu\right|\right|_{L_2(\partial Q)} ||Nu - Au||_{L_2(\partial Q)} + \\ &+ ||\gamma' Nu - w||\,||Nu - Au||. \end{aligned}$$

Moreover, $||Nu - Au||_{L_2(\partial Q)} \leq c||u||$, $||Nu - Au|| \leq ||Nu - Au||_{2,1} \leq c||u||$ and, by Proposition 4.1, for a given $\varepsilon > 0$ there is a $C_\varepsilon > 0$ such that

$$||\partial Nu/\partial n + \beta' Nu|\,|_{L_2(\partial Q)} \leq c((a_1 + \varepsilon)\,||Nu||L_2(\partial Q) + C_\varepsilon) \leq c((a_1 + \varepsilon)\,||u|| + C_\varepsilon).$$

Hence, there exists an $R_\varepsilon > 0$ such that for each $||u|| \geq R_\varepsilon$

$$\begin{aligned} &||Nu - Au||_{2,1}^2 \leq C^2 ||u||((a_1 + \varepsilon)\,||u|| + C_\varepsilon) + C^2\,||u||\,((a_2 + \varepsilon)u + C'_\varepsilon) \\ &\leq C^2(a_1 + a_2 + 3\varepsilon)\,||u||^2, \text{ where } a_1 = |\beta - \beta'| \text{ and } a_2 = |\gamma - \gamma'|. \quad \square \end{aligned}$$

Remark 4.1. Theorems 4.3 and 4.4 extend the corresponding results of Dias-Hernandez [3] involving asymptotically zero maximal monotone graphs β and γ, i.e. $|\beta-\beta'|=0$, $|\gamma-\gamma'|=0$, respectively. Their proofs are based on the generalized first Fredholm theorem of Nečas [11] for compact asymptotically zero mappings.

REFERENCES

[1] H. Brézis, *Probèmes unilatéraux*, J. Math. Pure Appl. **51** (1972), 1–168.

[2] H. Brézis, *Monotonicity methods in Hilbert spaces and some applications to nonlinear partial differential equations*, Nonlinear Functional Analysis, E. H. Zarantonello ed., Acad. Press, 1971, 101–156.

[3] J. P. Dias, J. Hernandez, *Bifurcation a l'inifini et alternative de Fredholm pour certains problemes unilateraux*, J. Math. Pure Appl. 55 (1976), 189–206.

[4] D. G. de Figueiredo, *The Dirichlet Problem for Nonlinear Elliptic Equations: A Hilbert Space Approach*, Lecture Notes in Maths., Vol. 466, Springer-Verlag, Berlin, 1975.

[5] P. M. Fitzpatrick, *On nonlinear perturbations of linear second-order elliptic boundary value problems*, Math. Proc. Cambridge Phil. Soc. **84** (1978), 143–157.

[6] R. I. Kachurovski, *On linear operators whose ranges are subspaces*, Dokl. Akad. Nauk SSSR **196** (1971), 168–172.

[7] J. L. Kazdan, F. W. Warner, *Remarks on some quasilinear elliptic equations*, Comm. Pure Appl. Math. **28** (1975), 567–597.

[8] P. S. Milojević, *Fredholm alternatives and surjectivity results for multivalued A-proper and condensing mappings with applications to nonlinear integral and differential equations*, Czech. Math. J. **30** (**105**) (1980), 387–417.

[9] P. S. Milojević, *Fredholm theory and semilinear equations without resonance involving noncompact perturbations*, I

[10] P. S. Milojević, *Solvability of nonlinear wave equations with nonmonotone nonlinearities*, (in preparation).

[11] J. Nečas, *Fredholm alternative for nonlinear operators and applications to partial differential equations and integral equations*, Čas. Pest. Mat. **9** (1972), 15–21.

[12] W. V. Petryshyn, *On the approximation-solvability of equations involving A-proper and pseudo A-proper mappings*, Bull. Amer. Math. Soc. **81** (1975), 223–312.

Department of Mathematics
New Jersey Institute of Technology
Newark, N. J., 07102, U. S. A.

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