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KÖTHE-TOEPLITZ DUALS OF SOME NEW SEQUENCE SPACES AND THEIR MATRIX MAPS

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Abstract. The spaces of sequences obtained by Kizmaz have been extended for a sequence $p = (p_k)$ of strictly positive numbers, and their Köthe-Toeplitz duals have also been obtained. Furthermore, we study the matrix transformations of these sequence spaces.

1. Introduction

Let l_∞, c and c_0, denote the Banach spaces of bounded, convergent, and null sequences $x = (x_k)$ respectively with $||x|| = \sup_k |x_k|$.

Recently Kizmaz [1] defined the sequence spaces

$$l_\infty(\Delta) = \{x = (x_k) : \Delta x \in l_\infty\}, \quad c(\Delta) = \{x = (x_k) : \Delta x \in c\},$$
$$c_0(\Delta) = \{x = (x_k) : \Delta x \in c_0\},$$

where $\Delta = (x_k - x_{k+1})$. These are Banach spaces with the norm $||x||_\Delta = |x_1| + ||\Delta x||$. For convenience we denote these spaces by Δl_∞, Δc, and Δc_0 and call the constituent sequences Δ-bounded, Δ-convergent, and Δ-null sequences respectively.

Let E be any of the spaces l_∞, c, c_0. Then it is easy to see that $E \subset \Delta E$. For example, let $x_k = k$, $k = 1, 2, \dots$. Then the sequence (x_k) is not convergent but it is Δ-convergent

In this paper, we extend these Δ-sequence spaces to $\Delta l_\infty(p)$, $\Delta c(p)$, and $\Delta c_0(p)$ for $p = (p_k)$ with $p_k > 0$, just like l_∞, c, and c_0 were extended to $l_\infty(p)$, $c(p)$, and $c_0(p)$ (Madox [4], Simons [5]). We also investigate their Köthe-Toeplitz duals and their matrix transformations.

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Let $p = (p_k)$ denote a sequence of strictly positive numbers (not necessarily bounded). We define

$$\Delta l_\infty(p) = \{x = (x_k) : \Delta x \in l_\infty(p)\}, \quad \Delta c(p) = \{x = (x_k) : \Delta x \in c(p)\},$$
$$\Delta c_0(p) = \{x = (x_k) : \Delta x \in c_0(p)\},$$

When (p_k) is constant with all terms equal to $p > 0$ we have $\Delta l_\infty(p) = l_\infty(\Delta)$, $\Delta c(p) = c(\Delta)$ and $\Delta c_0(p) = c_0(\Delta)$.

Let s be the space of all sequences (real or complex) and E a nonempty subset of s. Then we denote by E^+ the Kothe-Toeplitz dual of E, i. e.

$$E^+ = \{a = (a_k) : \textstyle\sum_k |a_k x_k| < \infty \text{ for every } x \in E\}.$$

Let $\Delta E(p)$ be any of the sets $\Delta l_\infty(p)$, $\Delta c(p)$, $\Delta c_0(p)$. Then we write $\Delta E(p;1)$ for $\Delta E^+(p)$, and $\Delta E(p;2)$ for $\Delta E^{++}(p)$ etc.

Throughout the paper, we shall write

$$\sup_k := \sup_{k=1,2,\dots}, \quad \lim_k := \lim_{k\to\infty}, \quad \textstyle\sum_k := \displaystyle\sum_{k=1}^{\infty}.$$

2. Köthe-Toeplitz duals

THEOREM 2.1. *For every sequence $p = (p_k)$, we hawe $\Delta l_\infty(p;1) = D_1(p)$, where*

$$D_1(p) = \cap_{N>1}\{a = (a_k) : \textstyle\sum_k k|a_k| N^{1/p_k} < \infty\}.$$

Proof. Let $a \in D_1(p)$ and $x \in \Delta l_\infty(p)$. Choose $N > \max(1, \sup_k k^{-1}|x_k|)$ Then

$$\textstyle\sum_k |a_k x_k| \le \sum_k k|a_k| \,(|x_k|/k) \le \sum_k k|a_k| N^{1/p_k} < \infty.$$

Therefore $a \in \Delta l_\infty(p;1)$.

Now suppose that $a \in \Delta l_\infty(p;1)$ but $a \notin D_1(p)$, i. e. there is an integer $N > 1$ such that $\sum_k k\,|a_k|\, N^{1/p_k} = \infty$. Choose $x = (x_k)$ with $x_k = kN^{1/p_k} \operatorname{sgn} a_k$. Then $x \in \Delta l_\infty(p)$ but $\sum_k |a_k x_k|$ is divergent. Hence $\Delta l_\infty(p;1) = D_1(p)$.

Remark. Similarly, we can define, for every $p = (p_k)$

$$\Delta c_0(p;1) = \cup_{N>1}\{a : \textstyle\sum_k k|a_k| N^{-1/p_k} < \infty\}.$$

THEOREM 2.2. *For every $p = (p_k)$, we have $\Delta l_\infty(p;2) = D_2(p)$, where*

$$D_2(p) = \cup_{N>1}\{a : \sup_k\,(k^{-1}|a_k|)\, N^{-1/p_k} < \infty\}.$$

Proof. Let $a \in D_2(p)$. Therefore, there is $N > 1$ such that for some constant B

$$\sup_k k^{-1}\,|a_k|\, N^{-1/p_k} \le B < \infty,$$

that is, $k^{-1}B^{-1}|a_k|N^{-1/p_k} \leq 1$ or $|a_k| \leq kBN^{1/p_k}$. Now,

$$\textstyle\sum_k |a_k x_k| = \sum_k |a_k||x_k| \leq \sum_k k^{-1}|a_k| \sum_k k|x_k| \leq B\sum_k k|x_k|N^{1/p_k} < \infty.$$

Therefore $a \in \Delta l_\infty(p;2)$.

Suppose $a \in \Delta l_\infty(p;2)$ and $a \notin D_2(p)$. Then there is a strictly increasing sequence (k_N) of positive integers such that for $k = k_N$, $|a_k|\, N^{-1/p_k} > N^2$. Now, define $x = (x_k)$ by

$$x_k = \begin{cases} 0, & k \neq k_N, \\ kN^{-(2+1/p_k)}\operatorname{sgn} a_k, & k = k_N. \end{cases} \qquad (N \geq 2)$$

Then, for every integer $N > 1$, we have $(k = k_N)$

$$|x_k|\, M^{1/p_k} \leq kN^{-2}$$

for every $N \geq M$. But, for $k = k_N$, $a_k x_k > 1$, which contradicts the fact that $a \in \Delta l_\infty(p;2)$.

Remark. It is easy to see that, for every $p = (p_k)$,

$$\Delta c_0(p;2) = \cap_{N>1}\{a : \sup_k (k^{-1}|a_k|)N^{1/p_k} < \infty\}.$$

THEOREM 2.3. *$\Delta l_\infty(p)$ is perfect if and only if $p \in l_\infty$.*

The proof is trivial (see [3]).

3. Some matrix mappings

Let X, Y be two nonempty subsets of the space s of all sequences and $A = (a_{nk})$ an infinite matrix, n, $k = 1, 2, \ldots$. For every $x = (x_k) \in X$ and every integer n we write $A_n(x) = \sum_k a_{nk}x_k$. The sequence $Ax = (A_n(x))$, if it exists, is called the transformation of x by the matrix A. We say $A \in (X, Y)$ if and only if $Ax \in Y$, whenever $x \in X$.

THEOREM 3.1. *Let $1 < p < \infty$. Then, $A \in (l_p, c(\Delta))$ if and only if there exists all integer $N > 1$ such that*

(i) $M = \sup_n \sum_{k=N}^{\infty} N|a_{nk}|^q < \infty \quad (1/p + 1/q = 1)$,

(ii) $\lim_n \Delta a_{nk} = a_k$ *exists for each fixed k, where*

$$\Delta a_{nk} = a_{nk} - a_{n+1,k}, \quad l_p = \{(x_k) : \textstyle\sum_k |x_k|^p < \infty\}.$$

Proof. Sufficiency. Suppose the conditions (i) and (ii) hold and that $x \in l_p$ Choose a fixed integer $n_0 \geq 1$ such that

$$\left(\sum_{k=n_0+1}^{\infty} |x_k|^p\right)^{1/q} \leq M^{1/q}\,\varepsilon/4,$$

Now,

$$|A_n(x) - A_{n+1}(x)| = \left|\sum_k (a_{nk} - a_{n+1,k})x_k\right|$$
$$\leq \sum_{k=1}^{n_0} |a_{nk} - a_{n+1,k}||x_k| + \sum_{k=n_0+1}^{\infty} |a_{nk}||x_k| + \sum_{k=n_0+1}^{\infty} |a_{n+1,k}||x_k|$$
$$\leq \left(\sum_{k=1}^{n_0} |a_{nk} - a_{n+1,k}|^q\right)^{1/q} \left(\sum_{k=1}^{n_0} |x_k|^p\right)^{1/p} + \varepsilon/2.$$

By virtue of (ii), we have $\Delta a_{nk} \in c$, so there is an integer m_0, such that

$$|a_{nk} - a_{n+1,k}| < \frac{\varepsilon}{2} \left(\sum_{k=1}^{n_0} |x_k|^p\right)^{1/p} (n_0)^{1/q}, \quad (n \geq n_0)$$

Hence $|A_n(x) - A_{n+1}(x)| \leq \dfrac{\varepsilon}{2} + \dfrac{\varepsilon}{2} = \varepsilon$.

Necessity. Condition (ii) follows from the fact that $e^k \in l_p$, where $e^k = (0, 0, \dots, 0, 1\,(k\text{-th spot}), 0, \dots)$. Now, let $T(x) = \sup_n |A(\Delta x)|$. Then T is a continuous seminorm on l_p and bounded too. By uniform boundedness principle, there is a constant L such that

$$(*) \qquad T(x) \leq L||x||.$$

Put $x = (x_k)$ with $x_k = 1$ $(k \geq N)$, 0 $(k < N)$ in $(*)$. Therefore (ii) holds true.

Theorem 3.2. *Let $0 < p \leq 1$. Then $A \in (l_p, c(\Delta))$ if and only if*
(i) $\sup_{n,k} |a_{nk}| < \infty$ (ii) $a_{nk} \to \alpha_k$ $(n \to \infty)$ *for each fixed k.*

Proof. Omitted.

Theorem 3.3. *Let $p_k > 0$ for every k. Then $A \in (\Delta l_\infty(p), l_\infty)$ if and only if, for every integer $N > 1$,*

$$\sup_n \textstyle\sum_k k|a_{nk}|N^{1/p_k} < \infty.$$

Proof. Sufficiency. Let the condition hold and $x \in \Delta l_\infty(p)$. Then

$$|A_n(x)| = |\textstyle\sum_k a_{nk}x_k| \leq \sum_k |a_{nk}|\,|x_k|$$
$$\leq \textstyle\sum_k k|a_{nk}|N^{1/p_k} \leq \sup_n \sum_k k|a_{nk}|N^{1/p_k} < \infty$$

for an integer $N > \max(1, \sup_k k^{-1}|x_k|^{p_k})$. Therefore $A \in (\Delta l_\infty(p), l_\infty)$.

Necessity. Suppose for an integer $N > 1$

$$\sup_n \textstyle\sum_k k|a_{nk}|N^{1/p_k} = \infty,$$

i. e. $(ka_{nk}N^{1/p_k}) \notin (l_\infty(\Delta), l_\infty)$. Therefore, there exists $x \in l_\infty(\Delta)$ with $\sup_k |x_k - x_{k+1}| = 1$ such that

$$\sup_n \sum_k k|a_{nk}||x_k - x_{k+1}|N^{1/p_k} = \infty.$$

Therefore, although $y = (N^{1/p_k}\Delta x_k) \in l_\infty(p)$, the sequence $(A_n(y)) \notin l_\infty$ which contradicts the fact that $A \in (\Delta l_\infty(p), l_\infty)$. This completes the proof.

THEOREM 3.4. *Let $p_k > 0$ for every k. Then $A(\Delta l_\infty(p), c)$ if and only if*
(i) $\sum_k k|a_{nk}|N^{1/p_k}$ *converges uniformly in n, for all integers $N > 1$, and*
(ii) $a_{nk} \to \alpha_k$ $(n \to \infty)$ *for each k.*

Proof. Omitted (see Lascarides and Maddox [2]).

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