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SOME LIMIT THEOREMS FOR ONE TYPE OF STOCHASTIC INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. We consider limit theorems for linear stochastic integro-differential equations of a special type and we give sufficient conditions for the almost sure convergence of the sequence of their solutions.

1. Introduction

Let us introduce some assumptions. Let (Ω, F, P) be a complete probability space on which an R^m-valued standard Wiener process $W = [(W_t, F_t),\ t \geq 0]$ is given, where $(F_t,\ t \geq 0)$ is a filtration satisfying the usual conditions. The non-random functions

$$a_n : [0,T] \times R^d \to R^d, \quad b_n : J \times R^d \to R^d, \quad c_n : J \times R^d \to R^d \times R^m,$$
$$A_n : [0,T] \times R^d \to R^d \times R^m, \quad B_n : J \times R^d \to R^d \times R^m,$$
$$C_n : J \times : R^d \to R^d \times R^m \times R^m,$$

where $T =$ const > 0, $J = \{(t,s) : (t,s) \in [0,T] \times [0,T], \leq t\}$, $n = 0, 1, \dots,$ are Borel-measurable in respect to the corresponding σ-fields on their domains. They satisfy the uniform Lipschitz conditions and the restriction on linear growth i.e. there exists a constant $K > 0$, such that for all $(t,s) \in J$, $x \in R^d$, $y \in R^d$,

$$|a_n(t,x) - a_n(t,y)| \leq K|x-y|, \ \ |C_n(t,s,x) - C_n(t,S,y)| \leq K|x-y|, \tag{1}$$

$$|a_n(t,x)^2 \leq K^2(1+|x|^2), \quad |C_n(t,s,x)^2 \leq K^2(1+|x|^2), \tag{2}$$

and analogously for the other functions.

We suppose that R^d-valued random processes $\varphi_n = (\varphi_n(t), t \in [0,T])$, $n = 0, 1, \dots,$ are nonanticipating for every $t \in [0,T]$. Also, in what follows we suppose that all integrals, ordinary and stochastic, exist as nonanticipating stochastic processes.

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Berger [1] considered a sequence of stochastic integro-differential equations (shortly SIDE) of the type

$$
\begin{aligned}
(3)\qquad & X_n(t) = \varphi_n(t)+ \\
&+\int_0^t [a_n(s,X_n(s)) + \int_0^s b_n(s,u,X_n(u))du + \int_0^s c_n(s,u,X_n(u))dW(u)]ds+ \\
&+\int_0^t [A_n(s,X_n(s)) + \int_0^s B_n(s,u,X_n(u))du + \int_0^s C_n(s,u,X_n(u))dW(u)]dW(s), \\
& n=0,1,\dots,
\end{aligned}
$$

and, under the conditions (1) and (2), he proved the existence and uniqueness of the R^d-valued, nonanticipating solutions $X_n = (X_n(t),\, t\in[0,T])$, $n=0,1,\dots.$ Also, he gave the conditions for the convergence in mean square of the sequence of solutions $\{X_n\}$, $n=1,2,\dots$, to the solution X_0 as $n\to\infty$.

The main problem of this paper is to give sufficient conditions under which the sequence of solutions $\{X_n\}$, $n=1,2,\dots$, converges almost surely to the solution X_0 as $n\to\infty$. We must require a closeness in some sense of the processes φ_n, $n=1,2,\dots$, and the functions a_n,b_n,c_n,A_0,B_0,C_0, $n=1,2,\dots$, to the process φ_0 and the functions a_0,b_0,c_0,A_0,B_0,C_0, respectively. The paper [5] contains some ideas and results about this closeness.

Let us introduce the following conditions:

$$
(4)\qquad \sup_n \sup_t E\{|\varphi_n(t)|^2\} < \infty;
$$

$$
(5)\qquad \sum_{n=1}^{\infty} E\{\sup_t |\varphi_n(t)-\varphi_0(t)|^2\} < \infty;
$$

$$
\begin{aligned}
(6)\qquad \sum_{n=1}^{\infty} \sup_{(t,s,x)\in\Pi} \{&|a_n(t,x)-a_0(t,x)| + |b_n(t,s,x)-b_0(t,s,x)|+ \\
&+|c_n(t,s,x)-c_0(t,s,x)| + |A_n(t,x)-A_0(t,x)|+ \\
&+|B_n(t,s,x)-B_0(t,s,x)| + |C_n(t,s,x)-C_0(t,s,x)|\} < \infty,
\end{aligned}
$$

where $\Pi = \{(t,s,x) : (t,s)\in J,\, x\in R\}$.

2. Main results

THEOREM 1. *Let the functions* a_n,b_n,c_n,A_n,B_n,C_n, $n=0,1,\dots,$ *and the processes* φ_n, $n=0,1,\dots,$ *be defined as above and all preceding conditions be satisfied. Then the sequence of random processes* $\{X_n\}$, $n=1,2,\dots,$ *converges almost surely, uniformly in* $t, t\in[0,T]$, *to the random process* X_0 *as* $n\to\infty$.

The conditions of Theorem 1 could be weakened. Following the tradition of the classical theory of stochastic differential equations, we suppose that for each number $M>0$ there exists a constant $L_M>0$, such that the condition (1) is valid with this constant for all $(t,s)\in J$, $|x|\le M$, $|y|\le M$. Also, the expectation in (4) does not have to be bounded, and (2) and (5) are satisfied.

THEOREM 2. *Let the functions* $a_n, b_n, c_n, A_n, B_n, C_n$, $n = 0, 1, \ldots,$ *and the processes* φ_n, $n = 0, 1, \ldots,$ *satisfy the preceding conditions and let (6) be valid for* $\Pi' = \{(t,s,x) : (t,s) \in J, |x| \leq M\}$ *instead of* Π. *Then the sequence of random processes* $\{X_n\}$, $n = 1, 2, \ldots,$ *converges almost surely, uniformly in* $t, t \in [0,T]$, *to the random process* X_0 *as* $n \to \infty$.

3. Proofs of theorems

Proof of Theorem 1. Denote

$$\begin{aligned}
&\varepsilon_n = E\{\sup_{(t,s)\in J}[|a_n(s,X_n(s)) - a_0(s,X_n(s))|^2 + \\
(7)\quad &+ |b_n(t,s,X_n(s)) - b_0(t,s,X_n(s))|^2 + |c_n(t,s,X_n(s)) - c_0(t,s,X_n(s))|^2 + \\
&+ |A_n(s,X_n(s)) - A_0(s,X_n(s))|^2 + |B_n(t,s,X_n(s)) - B_0(t,sS,X_n(s))|^2 + \\
&+ |C_n(t,s,X_n(s)) - C_0(t,s,X_n(s))|^2\},\ \ n = 1,2,\ldots,
\end{aligned}$$

$$\delta_n = E\{\sup t|\varphi_n(t) - \varphi_0(t)|^2\}, \quad n = 1, 2, \ldots \tag{8}$$

Since we must find an upper bound for $E\{|X_n(t) - X_0(t)|^2\}$, we will estimate only one integral, and analogously the others. If we apply one of the basic properties of the Ito integrals (see [2], [3]), add some terms, use the Cauchy-Schwartz inequality, the condition (1) and the notation (7), we obtain

$$\begin{aligned}
&E\left\{\int_0^t\int_0^s [B_n,(s,u,X_n(u)) - B_0(s,u,X_0(u))]dudW(s)\right\}^2 = \\
&= \int_0^t E\left\{\Big|\int_0^s [B_n(s,u,X_n(u)) - B_0(s,u,X_0(u))]du\Big|^2\right\}ds \leq \\
(9)\quad &\leq 2\int_0^t s\int_0^s E\{|B_n(s,u,X_n(u)) - B_0(s,u,X_n(u))|^2 + \\
&+ |B_0(s,u,X_n|(u) - B_0(s,u,X_0))|^2\}duds \leq \\
&\leq t^2[E\varepsilon t + K^2 + \int_0^t E\{|X_n(s) - X_0(s)|^2\}ds.
\end{aligned}$$

Subtracting the equations (3) for $n = 0$ from (3) and using (8), it is easy to obtain the estimation

$$\{E|X_n(t) - X_0(t)|^2\} \leq 7\delta_n + \alpha\left[\varepsilon_n T + K^2\int_0^t E\{|X_n(s) - X_0(S)|^2\}ds\right],$$

where $\alpha = 14[T^3/2 + 3T^2/2 + 2T + 1]$. By the Gronwall-Bellman inequality we have

$$E\{|X_n(t)l - X_0(t)|^2 \leq (7\delta_n + \alpha\varepsilon_n T)e^{\alpha K^2 t}. \tag{10}$$

We will apply the well-known inequality for stochastic Ito integrals:

$$E\Big\{\sup_{t:t\in[0,T]}\int_0^t X(s,\omega)dW(s)\Big\}^2 \le 4\int_0^T E\{|X(t,\omega)|^2\}dt,$$

if the last integral is finite (see [2], [3]).

For example, from (9) it follows that

$$\Big\{E\sup_{t:t\in[0,T]}\int_0^t\int_0^s [B_n(s,u,X_n(u))-B_0(s,u,X_0(u))]dudW(s)\Big\}^2 \le$$
$$\le 4\int_0^T E\Big\{\Big|\int_0^s [B_n(s,u,X_n(u))-B_0(s,u,X_0(u))]du\Big|^2\Big\}ds \le$$
$$\le 4T^2\Big[\varepsilon T + K^2\int_0^T E\{X_n(s)-X_0(s)|^2\}ds\Big].$$

We can find upper bounds for the other integrals similarly. So we have

$$E\{\sup_{t:t\in[0,T]}|X_n(t)-X_0(t)|^2\} \le 7\delta_n + \beta\Big[\varepsilon_n T + K^2\int_0^T E\{|X_n(t)-X_0(t)|^2\}dt\Big],$$

where $\beta = 14[T^3/2+3T^2+5T+4]$. Thus, from (10) and the last relation, it follows that

$$E\{\sup_{t:t\in[0,T]}|X_n(t)-X_0(t)|^2\} \le c_1\delta_n + c_2\varepsilon_n,$$

where c_1 and c_2 are corresponding constants. By Chebyshev's inequality, for each $\varepsilon > 0$ we have

$$\sum_{n=1}^{\infty} P\{\sup_{t:t\in[0,T]}|X_n(t)-X_0(t)| \ge \varepsilon\} \le \frac{1}{\varepsilon^2}\sum_{n=1}^{\infty} E\{\sup_{t:t\in[0,T]}|X_n(t)-X_0(t)|^2\} \le$$
$$\le \frac{c_1}{\varepsilon^2}\sum_{n=1}^{\infty}\delta_n + \frac{c_2}{\varepsilon^2}\sum_{n=1}^{\infty}\varepsilon_n < \infty,$$

because, from the conditions (5) and (6), the series $\sum_{n=1}^{\infty}\delta_n$ and $\sum_{n=1}^{\infty}\varepsilon_n$ are convergent. By the Borel-Cantelli's lemma and Weierstrass' uniform convergence theorem, it follows that the sequence of random processes $\{X_n\}, n = 1,2,\dots,$ converges almost surely, uniformly in $t, t \in [0,T]$, to the random process X_0 as $n \to \infty$. Thus Theorem 1 is proved.

Proof of Theorem 2. For a chosen number $M > 0$, let us denote

$$\psi_M(x) = \begin{cases} x, & \text{if } |x| < M \\ M\operatorname{sgn} x & \text{if } |x| \ge M \end{cases}$$

$$\varphi_n^M(t) = \psi_M(\varphi_n(t)), a_n^M(t,x) = a_n^M(t,\psi_M(x)), b_n^M(t,s,x) = b_n^M(t,s,\psi_M(x)),$$

and analogously for c_n, A_n, B_n, C_n, $n = 0, 1, \ldots$. Let $X_n^M(t)$ be a solution of the SIDE

$$X_n^M(t) = \varphi_n^M(t) +$$
$$+ \int_0^t \left[a_n^M(s, X_n^m(s)) + \int_0^s b_n^M(s, u, X_n^M(u))du + \int_0^s c_n^M(s, u, X_n^M(u))dW(u) \right] ds +$$
$$+ \int_0^t \Big[A_n^M(s, X_n^M(s)) + \int_0^t B_n^M(s, u, X_n^M(u))du +$$
$$+ \int_0^s C_n^M(s, u, X_n^M(u))dW(u) dW(s), \quad n = 0, 1, \ldots$$

These solutions exist by the existence and uniqueness theorem. Also, all conditions of Theorem 1 are satisfied and thus the sequence $\{X_n^M\}$ $n = 1, 2, \ldots,$ converges almost surely to the process X_0 as $n \to \infty$. From that fact, we will show that $\{X_n\}$, $n = 1, 2, \ldots,$ converges almost surely to X_0 as $n \to \infty$. Let

$$\tau_n^M = \begin{cases} \inf\{t : |X_n^M(t)| > M\} \\ T, \text{ if } |X_n(t)| \geq M \text{ for all } t \end{cases}$$

be stopping times with respect to $(F_t, t \geq 0)$. For each t, $t \in [0, T]$, we can find a sufficiently large M, such that $\tau_n^M > t$, $n = 0, 1, \ldots,$ almost surely. Since there exists a stopping time $\tau^M = \in_n \tau_n^M$, (see [4]) then on the interval $[0, \tau^M]$ we have

$$\varphi_n^M(t) = \varphi_n(t), \;\; a_n^M(t, X_n^M(t)) = a_n(t, X_n(t)), \;\; b_n^M(t, s, X_n^M(s)) = b_n(t, s, X_n(s)),$$

and similarly for c_n, A_n, B_n, C_n, $n = 0, 1, \ldots$. Thus on the interval $[0, \tau_M]$ the sequence $\{X_n\}$ $n = 1, 2, \ldots,$ converges almost surely to X_0 as $n \to \infty$. Since

$$\lim_{M \to \infty} P\{\tau^M = T\} = 1$$

(see [1]), it follows that the sequence $\{X_n\}$, $n = 1, 2, \ldots,$ converges almost surely, uniformly in t, $t \in [0, T]$, to X_0 as $n \to \infty$. Thus the proof is complete.

Remark. Theorem 1 and Theorem 2 can be proved if the coefficients of the SIDE (3) are random functions. In this case, a_n and A_n must be nonanticipating in s for each x, and b_n, B_n, c_n, C_n, must be nonanticipating in s for all (t, x). Also, the conditions (1), (2) and (6) must be satisfied almost surely.

REFERENCES

[1] M. Berger, *Stochastic Ito-Voltera Equations*, (Ph. D. Thesis). Carnegie-Mellon University, Pittsburgh, 1977.

[2] A. Friedman, *Stochastic Differential Equations and Applications, Vol. 1*, Academic Press, New York, 1975.

[3] I.I. Gihman, A.V. Skorohod, *Stochastic Differential Equations and Applications*, Naukova Dumka, Kiev, 1982. (In Russian).

[4] E. Wong, *Stochastic Processes in Information and Dynamical Systems*, Mc. Graw-Hill, New York, 1971.

[5] S. Janković, *One approximation of solution of stochastic differential equation in Differential Equations and Applications*, (Proc. 3rd Conf., Rousse '85), Rousse, Bulgaria, 1986.

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