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A CHARACTERISTIC PROPERTY FOR MAXIMA OF RANDOM VARIABLES WITH TRIVIAL ANALOGUE WHEN SUMS ARE CONCERNED

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Abstract. A class of random variables obtained by the operation of thinning is investigated and it is shown to belong to degenerate type.

Let X be a random variable and let $F(x)$ and $f(t)$ be, respectively, its distribution and its characteristic function. Characteristic function $f(t)$, being the Fourier-Stieltjes transform of $F(x)$, is well known to be connected with $F(x)$ by certain sort of "duality". Namely, to the operation of summing independent random variables, where characteristic functions of the addends are multiplied, corresponds the operation of taking extreme of independent random variables, where, in turn, the distribution functions are multiplied. Various well known results in probability theory could be understood as "dual" in the mentioned sense (for example the classes of stable and extremal distributions and related results). Here, we shall investigate a class of distributions obtained by the operation of thinning the sums of independent identically distributed random variables, whose dual was investigated in [1].

Let $X_1, X_2, \dots, X_k, \dots$ be independent and identically distributed random variables with the distribution function $F(x)$. Let us form the sums

$$Y_1 = X_1, Y_2 = X_1 + X_2, \dots, Y_k = X_1 + X_2 + \cdots + X_k, \dots$$

We perform the operation of thinning. Namely for given $p (0 < p < 1)$, we eliminate (independently of the others) the sums Y_k with probability p. Let us denote by $F_p(x)$ the distribution function of the first retained sum. Then we have

$$F_p(x) = \sum_{k=1}^{\infty} p^{k-1} q F^{*(k)}(x) \tag{1}$$

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where $q = 1-p$ and $F^{*(k)}(x)$ is distribution function of Y_k. In terms of characteristic functions the equality (1) is equivalent to

$$f_p(t) = q\sum_{k=1}^{\infty} p^{k-1}(f(t))^k = \frac{qf(t)}{1-(1-q)f(t)},$$

where $f_p(t)$ and $f(t)$ are characteristic functions of distributions $F_p(x)$ and $F(x)$.

Let us denote by D the class of distribution functions whose members fulfill the following condition: for every $p, q(p+q=1,\ 1>p>0)$, real functions $a(q)>0$ and $b(q)\not\equiv 0$ exist, such that characteristic function $f_p(t)$ of the first retained sum satisfies the following equality:

$$f_p(t) = f(a(q)t + b(q)).$$

THEOREM. *Distribution function $F(x)$ belongs to the class D if and only if it is degenerate at zero.*

Proof. If distribution function $F(x)$ is degenerate at zero, it could be easily checked that its characteristic function $f(t)=1$ fulfills the equation (2)

$$\frac{qf(t)}{1-(1-q)f(t)} = f(a(q)t + b(q)), \tag{2}$$

$a(q)>0, b(q)\not\equiv 0$, for every $q(0<q<1)$ and every t.

Now let prove the converse proposition: if $F(x)$ belongs to the class D, then it must be degenerate.

By assumption (2) is true. Let $t=0$, then it follows from (2) that $f(b(q))=1$. If $t=b(q)\neq 0$, we have

$$1 = \frac{qf(b(q))}{1-(1-q)f(b(q))} = f(a(q)b(q+b(q)).$$

By iterating the previous procedure, we obtain that

$$f(b(q)(a^n(q) +^{n-1}(q) + \cdots + a(q)+1)) = 1$$

is valid for every $n=1,2,\ldots$. In other words, for every t_i of the form

$$t_i = b(q)(a^n(q) + a^{n-1}(q) + \cdots + a(q)+1)), \tag{3}$$

we have that $f(t_i)=1$. Therefore [2] $f(t)$ is the characteristic function of purely discrete probability distribution $F(x)$. Since $f(t_i)=1$ for $i=0,1,\ldots$, it follows [2] that points of discontinuity of probability distribution $F(t)$ are contained in the set of zeros of the function $1-\cos(t_i x)$, which means that discontinuity points of probability distribution $F(x)$ are of the form $2\pi k/t_i$, $k\in Z$, $i=0,1,\ldots$, or, more precisely, of the form:

$$\frac{2\pi k}{b(q)(a^i(q) + a^{i-1}(q) + \cdots + a(q)+1)} \quad k\in Z,\ \ i=0,1,\ldots$$

So we have that discontinuity points of probability distribution $F(x)$ are contained in the set M_n

$$M_n = \left\{ \frac{2\pi k}{b(q)(a^n(q) + a^{n-1}(q) + \cdots + a(q) + 1)} \quad k \in Z \right\},$$

for each $n = 0, 1, 2, \ldots$ and consequently, discontinuity points are contained in $\bigcap_n M_n$ too. It follows that if s is a point of discontinuity of probability distribution $F(x)$, then it must be of the form:

$$s = \frac{2\pi k_0}{b(q)} = \frac{2\pi k_1}{b(q)} = \frac{2\pi k_1}{(a(q)+1)} = \frac{2\pi k_2}{b(q)(a^2(q) + a(q) + 1)} = \ldots \tag{4}$$

for some $k_0, k_1, k_2, \dots \in Z$. In other words, since all discontinuity points are in the set $\bigcap_n M_n$, then for each n there exists some $k_n \in Z$, such that

$$s = \frac{2\pi k_n}{b(q)(a^n(q) + a^{n-1}(q) + \cdots + a(q) + 1)}.$$

Let us divide the equalities (4) by $2\pi/(q)$. We have:

$$k_0 = \frac{k_1}{a(q)+1} = \frac{k_2}{a_2(q) + a(q) + 1} = \ldots \tag{5}$$

From the first equality we obtain

$$a(q) = k_1/k_0 - 1. \tag{6}$$

It follows from (6) that $a(q)$ is a rational number, so we can write it in the following way: $a(q) = p_1/p_2$, $p_1, p_2 \in Z$ and p_1, p_2 are relatively prime. If we substitute $a(q)$ from (6) in (5), we get:

$$k_0 = \frac{k_1}{p_1/p_2 + 1} = \frac{k_2}{p_1^2/p_2^2 + p_1/p_2 + 1} = \ldots,$$

which is equal to

$$k_0 = k_1 p_2 p_1 + p_2 = \frac{k_2 p_2^2}{p_2^2 + p_1 p_2 + p_1^2} = \ldots$$

Since k_0 is integer, then $\dfrac{k_1 p_2}{p_1 + p_2}$ must be integer too. Since p_1 and p_2 are relatively prime then $p_1 + p_2$ and p_2 are relatively prime. Therefore k_1 is divisible by $p_1 + p_2$ and k_0 is divisible by p_2. In the same way k_0 is divisible by p_2^n for every $n = 1, 2, \ldots$. It follows that there must be $k_0 = 0$, which means that discrete probability distribution $F(x)$ has only one jump and it is located at zero.

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