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ON THE ZEROS OF A POLYNOMIAL

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Abstract. Using an extension of Hölder's inequality, we obtained an upper bound for the moduli of zeros of a polynomial with complex coefficients.

According to a classical result of Kuniyeda, Montel and Toya [2, p. 124], on the location of zeros, of a polynomial, all the zeros of the polynomial $P(z) = \sum_{k=0}^{n} a_k z^k$ $(a_n \neq 0)$ of degree n, lie in the circle

$$|z| < \left[1 + \left\{ \sum_{j=0}^{n=1} \left| \frac{a_j}{a_n} \right|^p \right\}^{q/p} \right]^{1/q} \tag{1}$$

where $p > 1$, $q > 1$, $1/p + 1/q = 1$.

Hölder's inequality was used to obtain the bound (1). In this note, we have used an extension of Hölder's inequality to obtain the following:

THEOREM. *All the zeros of the polynomial* $p(z) = \sum_{k=0}^{n} a_k z^k$, *of degree* n, *lie in the circle*

$$|z| < \aleph^{1/q}, \tag{2}$$

where $\aleph$ *is the unique root of the equation*

$$x^3 - (1 + DN)x^2 + DNx - D = 0, \tag{3}$$

in $(1, \infty)$,

$$D = \left\{ \sum_{j=0}^{n=1} |a_j|^p / |a_n|^p \right\}^{q/p},$$

$$N = (|a_{n-1}| + |a_{n-2}|)^q (|a_{n-1}^p + |a_{n-2}|^p)^{-(q-1)},$$

$$p > 1\, q > 1,\ 1/p + 1/q = 1.$$

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Remark 1. The bound (2) is better than the bound (1), for the polynomials, whose coefficients satisfy the inequality

$$D < (2-N)/(N-1), \tag{4}$$

as can be easily seen from the following:

$$\begin{aligned} &f(x) = x^3 - (1+DN)x^2 + DNx - D, \\ &f(x) > 0, \text{ for } x > K; f(x) < 0, \text{ for } 1 < x < K \\ &f(1+D) = (1+D)^3 - (1+DN)(1+D)^2 + DN(1+D) - D \\ &\qquad\quad = D^2[(2-N) - D(N-1)] > 0 \quad \text{(by (3))}, \end{aligned}$$

which implies $(1+D) > K$.

The polynomial $P(z) = a_0 + a_1 z + \cdots + a_5 z^5$ with $|a_0| = 1$, $|a_1| = 2$, $|a_2| = 1$, $|a_3| = 1$, $|a_4| = 4$, $|a_5| = 6$, satisfies (4), for $q = p = 2$.

Remark 2. It is quite easy to show that the equation (3) has only one real root in $(1, \infty)$.

Let

$$\begin{aligned} F(y) = f(1+y) &= (1+y)^3 - (1+DN)(1+y)^2 + DN(1+y) - D \\ &= y^3 + (2-DN)y^2 + (1-DN)y - D. \end{aligned}$$

Now, let us consider three different possibilities:

$$(i) \quad 0 < DN \leq 1; \qquad (ii) \quad 1 < DN \leq 2; \qquad (iii) \quad 2 < DN.$$

If we consider (i), then $F(y)$ has only one change of sign. Hence, according to Descartes' rule of signs, the equation $F(y) = 0$ will have at most one positive root. The same is also true for (ii) and (iii). Hence, whatever be the value of DN, the equation $F(y) = 0$ will have at most one positive root. Furthermore $F(0) = -D$; $F(y) \to +\infty$ as $y \to \infty$. Hence, the equation $F(y) = 0$ has exactly one positive root. This obviously implies that the equation $f(x) = 0$ i.e., the equation (3) has only one root in $(1, \infty)$.

For the proof of the theorem we require the following extension of Holder's inequality:

Lemma. (Beckenbach [1]) *Let* $\alpha_j > 0$, $\beta_j > 0$ *for* $j = 1, 2, \ldots, n$ *and* $p > 1$, $q > 1$ *with* $1/p + 1/q = 1$. *Then*

$$\begin{aligned} \sum_{j=1}^{n} \alpha_j \beta_j \leq &\left(\left(\sum_{j=1}^{n} \beta_j^p \right)^{1/p} \left(\sum_{j=1}^{m} \right)^{-1/p} \right) \cdot \\ &\cdot \left[\left(\sum_{j=1}^{m} \alpha_j \beta_j \right)^q + \left(\left(\sum_{j=1}^{m} \beta_j^p \right)^{(q-1)} \right) \left(\sum_{j=m+1}^{n} \alpha_j^q \right) \right]^{1/q} \end{aligned} \tag{4}$$

Proof of the theorem. For $|z| > 1$, we have

$$|P(z)| \geq |a_n|\,|z|^n - \sum_{j=0}^{n-1} |z|^j |a_i|$$

$$\geq |a_n|\,|z|^n - \left(\sum_{j=0}^{n-1} |a_j|^p\right)^{1/p} (|a_{n-1}|^p + |a_{n-2}|^p)^{-1/p}$$

$$\cdot \Bigg[(|a_{n-1}||z|^{n-1} + |a_{n-2}||z|^{n-2})^q + (|a_{n-1}|^p +$$

$$+|a_{n-2}|^p)^{(q-1)} \left(\sum_{j=0}^{n-1} |z|^{jq}\right)\Bigg]^{1/q}, \qquad \text{(by (5))}$$

$$\geq |a_n||z|^n - \left(\left(\sum_{j=0}^{n-1} |a_j|^p\right)^{1/p}\right) \cdot$$

$$\left\{\left(|z|^{(n-1)q}(|a_{n-1}| + \alpha_{n-2}|)^q (|a_{n-1}|^p + |a_{n-2}|^p)^{-q+1}\right) + \left(\sum_{j=0}^{n-3} |z|^{jq}\right)\right\}^{1/q}$$

$$= |a_n||z|^n \left[1 - D^{1/q}\left\{(N|z|^{-q}) + \left(\sum_{j=3}^{n} |z|^{-jq}\right)\right\}^{1/q}\right]$$

$$> |a_n||z|^n \left[1 - D^{1/q}\left\{(N|z|^{-q}) + \left(\sum_{j=3}^{\infty} |z|^{-jq}\right)\right\}^{1/q}\right]$$

$$= |a_n||z|^n[1 - D^{1/q}\{(N|z|^q(|z|^q - 1) + 1)/(|z|^{2q}(|z|^q - 1))\}^{1/q}]$$

$$= |a_n||z|^n[1 - \{(Dn|z|^{2q} - DN|z|^q + D)/(|z|^{3q} - |z|^{2q})\}^{1/q}]$$

$$\geq 0, \text{ if } 1 \geq (DN|z|^{2q} - DN|z|^q + D)/(|Z|^{3q} - |z|^{2q}).$$

So, we conclude that $P(z) \neq 0$, for $|z| > 1$, if

$$(|z|^q)^3 - (|z|^q)^2 - DN(|z|^q)^2 + DN(|z|^q) - D > 0.$$

Replacing $|z|^q$ by x, we get the result.

REFERENCES

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