INFLATION OF SEMIGROUPS

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Abstract. We introduce the concept of an $n$-inflation of a semigroup. In particular, for $n = 1$ we obtain the inflation introduced by Clifford [6], and for $n = 2$ the strong inflation introduced by Petrich [6]. We also characterize $n$-inflations of unions of groups, of semilattices of groups of unions of periodic groups, etc. In addition, we describe nilpotent semigroups of arbitrary nilpotency class.

1. Introduction and preliminaries

Let $S$ and $T$ be two disjoint semigroups and suppose that $T$ has a zero element. A semigroup $V$ is said to be an (ideal) extension of $S$ by $T$ if it contains $S$ as an ideal and the Rees factor semigroup $V/S$ is isomorphic to $T$. If, in addition, there is a partial homomorphism $\varphi : T \setminus 0 \to S$ such that for all $A, B \in T \setminus 0$ and $c, d \in S$:

$$A \cdot B = \begin{cases} AB, & \text{if } AB \neq 0 \text{ in } T \\ \varphi(A) \varphi(B), & \text{if } AB = 0 \text{ in } T \end{cases}$$

$$A \circ c = \varphi(A)c, \quad c \circ A = \alpha \varphi(A), \quad c \circ d = cd$$

we say that extension $V$ is determined by that partial homomorphism, [6].

Let $V$ be an extension of $S$. Then $V$ is a retract extension if there exists a homomorphism $\varphi$ of $V$ onto $S$ and $\varphi(x) = x$ for all $x \in S$. In this case we call $\varphi$ a retraction. Petrich [9] proved that an extension $V$ of a semigroup $S$ by a semigroup $T$ with zero is determined by a partial homomorphism if and only if $V$ is a retract extension of $S$. Here we give one more characterization of the retract extension.

Proposition 1.1. Let $T$ be a semigroup. With each $a \in T$ associate a set $Y_a$ such that

$$a \in Y_a, \quad Y_a \cap Y_b = \emptyset \text{ if } a \neq b.$$  

(1.1)

Let

$$\varphi^{(a,b)} : Y_a \times Y_b \to Y_{a \cdot b}$$

(1.2)

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\[ \varphi^{(a,b)}(x, b) = \varphi^{(a,b)}(a, y) = ab \text{ for all } x \in Y_a \text{ and } y \in Y_b \text{ be functions for which} \]

\[ \varphi^{(a,b,c)}(\varphi^{(a,b)}(x, y), z) = \varphi^{(a,b,c)}(x, \varphi^{(b,c)}(y, z)) \]

and define a multiplication \( \ast \) on \( S = \bigcup_{a \in T} Y_a \) by:

\[ x \ast y = \varphi^{(a,b)}(x, y) \quad \text{if } x \in Y_a, \; y \in Y_b. \]

Then \( (S, \ast) \) is a semigroup and \( S \) is a retract extension of \( T \). Conversely, every retract extension \( S \) of a semigroup \( T \) can be so constructed.

**Proof.** Suppose that \( S \) fulfills the conditions of the proposition. Let \( x \in Y_a, \; y \in Y_b, z \in T_c \). Then by (1.3) we have

\[
(x \ast y) \ast z = \varphi^{(a,b)}(x, y) \ast z = \varphi^{(a,b,c)}(\varphi^{(a,b)}(x, y), z) \\
= \varphi^{(a,b,c)}(\varphi^{(b,c)}(y, z), x) = \varphi^{(b,c)}(y, z) \ast \varphi^{(a,b)}(x, y) \\
= x \ast (y \ast z).
\]

Hence \( (S, \ast) \) is a semigroup. Define a mapping \( \varphi : S \to T \) by \( \varphi(Y_a) = a \). It is clear that \( \varphi \) is onto and that \( \varphi(a) = a \) for all \( a \in T \). Furthermore, for \( x \in Y_a, \; y \in Y_b \) we have

\[ \varphi(x \ast y) = \varphi(\varphi^{(a,b)}(x, y)) = ab = \varphi(x)\varphi(y). \]

Thus \( \varphi \) is a homomorphism and by (1.2) \( T \) is an ideal of \( S \). Therefore, \( S \) is a retract extension of \( T \).

Conversely, let \( S \) be a retract extension of \( T \). Then there is a homomorphism \( \varphi \) of \( S \) onto \( T \) such that \( \varphi(a) = \varphi^{-1}(a) \) for all \( a \in T \). For \( a \in T \) assume that \( Y_a \) is the set \( a \in T \) and the condition (1.1) is satisfied.

For any \( x, y \in S \) there exist \( a, b \in T \) such that \( x \in Y_a, \; y \in Y_b \), so that \( \varphi(x) = a, \; \varphi(y) = b \). From this it follows that

\[ \varphi(xy) = \varphi(x)\varphi(y) = ab \in Y_{ab} \]

i.e. \( xy \in Y_{ab} \). Hence there exist the functions

\[ \varphi^{(a,b)} : Y_a \times Y_b \to Y_{ab} \]

and it is clear that for these functions (1.3) holds. Since \( T \) is an ideal of \( S \) we have (1.2).

Clifford [6, p. 98] gave a construction for a special retract extension of a semigroup, the so-called inflation of a semigroup. A semigroup \( S \) is an inflation of a semigroup \( T \) if \( T \) is a subsemigroup of \( S \) and there is a mapping \( \varphi \) of \( S \) onto \( T \) such that \( \varphi(x) = x \) for \( x \in T \) and \( xy = \varphi(x)\varphi(y) \) for \( x, y \in S \). For further results concerning inflation of a semigroup, see [1], [3], [13], [14].

Petrich [10], [11], generalized Clifford's result introducing the notion of strong inflation.
Let $T$ be a semigroup. To each $a \in T$ we associate two sets $X_a$ and $Y_a$ having the following properties:

$$a \in X_a, \ X_a \cap X_b = Y_a \cap Y_b = \emptyset \text{ if } a \neq b \quad X_a \cap Y_b = \emptyset \ (a, b \in T).$$

To every pair of elements $x \in Y_a, y \in Y_b$, we associate an element $\varphi^{(a, b)}(x, y) \in X_a \cap Y_b$. Now let $Z_a = X_a \cup Y_a$ and define a multiplication $\ast$ on $S = \bigcup_{a \in T} Z_a$ by: if $x \in Z_a, y \in Z_b$, then

$$x \ast y = \begin{cases} 
\varphi^{(a, b)}(x, y) & \text{if } x \in Y_a, y \in Y_b \\
abla & \text{otherwise.}
\end{cases}$$

Then $S$ is a retract extension of $T$ and $S^3 \subset T$. Conversely, every retract extension $S$ of a semigroup $T$ such that $S^3 \subset T$ can be so constructed. Such a semigroup $S$ is called a strong inflation of a semigroup $T$. In particular for $T = 0$ nilpotent semigroups of nilpotency class $\leq 3$ are described, [12, p. 135]. Moreover, a semigroup $S$ in $n$-nilpotent if $S^n = 0 \ (n \in \mathbb{Z}^+)$. In this paper we introduce the notion of an $n$-inflation of a semigroup. For $n = 1$ we obtain the inflation and for $n = 2$ we obtain the strong inflation of semigroup. In Theorem 2.1, we describe an $n$-inflation of an arbitrary semigroup by means of retraction. In section 2, also, a description of a strong $n$-inflation is given (Theorem 2.2) and nilpotent semigroups of arbitrary nilpotency classes. In addition, we give characterizations of $n$-inflations of some special semigroups: unions of groups, semilattices of groups, unions of periodic groups and so on.

For undefined notions and notations we refer to [4], [6] and [12].

### 2. $n$-inflation of a semigroup

We introduce here the notion of an $n$-inflation of a semigroup.

**Lemma 2.1.** Let $T$ be a semigroup. To each $a \in T$ we associate a family of sets $X_i^a \ (i = 1, 2, \ldots, n)$ such that $a \in X_r^a$ for some $r \in \{1, 2, \ldots, n\}$ and

$$X_i^a \cap X_j^b = \emptyset \quad \text{if } i \neq j; \ X_i^a \cap X_j^b = \emptyset \quad \text{if } a \neq b. \tag{2.1}$$

Let, for nonempty sets $X_i^a$ and $X_j^b$,

$$\Phi^{(a, b)}_{(i, j)} : X_i^a \times X_j^b \to \bigcup_{\nu=i+j} X_{\nu}^{ab} \quad \text{if } i + j \leq n \tag{2.2}$$

$$\Phi^{(a, b)}_{(i, j)}(x, y) = \begin{cases} 
abla & \text{if } i + j > n \\
abla & \Phi^{(a, b)}_{(i, j)}(a, y) = \Phi^{(a, b)}_{(i, j)}(x, y) = \begin{cases} \Phi^{(a, b)}_{(i, j)}(x, y) & \text{if } i + j \leq n  \\
abla & \text{if } i + j > n \end{cases} \end{cases}$$

be functions for which:

$$\forall s \geq i + j \forall t \geq j + k \Phi^{(a, b)}_{(s, k)} \left( \Phi^{(a, b)}_{(i, j)}(x, y), z \right) = \Phi^{(a, b)}_{(i, j)} \left( x, \Phi^{(b, c)}_{(j, k)}(y, z) \right) \tag{2.3}$$
for all $a, b, c \in T$, where $i + j \leq n$ or $j + k \leq n$ or $i + t \leq n$ or $s + k \leq n$.

Let $Y_a = \bigcup_{i=1}^n X_i^a$ and define a multiplication $\ast$ on $S = \bigcup_{a \in T} Y_a$ by: for $x \in Y_a, y \in Y_b$,

$$x \ast y = \Phi^{(a, b)}_{(i, j)}(x, y) \quad \text{if} \quad x \in X_i^a, y \in X_j^b, 1 \leq i, j \leq n$$

Then $(S, \ast)$ is a semigroup.

**Proof.** Let $x, y, z \in S$. Then there exist $a, b, c \in T$ such that $x \in Y_a, y \in Y_b, z \in Y_c$ i.e. $x \in X_i^a, y \in X_j^b, z \in X_k^c$ for some $1 \leq i, j, k \leq n$. Assume that $i + j \leq n$ and $j + k \leq n$. Then

$$(x \ast y) \ast z = \Phi^{(a, b)}_{(i, j)}(x, y) \ast z, \quad \Phi^{(a, b)}_{(i, j)}(x, y) \in X_i^a, \quad i + j \leq s \leq n$$

$$= \Phi^{(a, c)}_{(s, k)} \left( \Phi^{(a, b)}_{(i, j)}(x, y), z \right) \quad \Phi^{(a, b)}_{(i, j)}(x, y, z) \in X_i^a, \quad j + k \leq t \leq n$$

$$= \Phi^{(a, c)}_{(i, t)} \left( x \Phi^{(b, c)}_{(j, k)}(y, z) \right)$$

and by (2.3) we have associativity. In other cases it can be, in a similar way, proved that the associativity holds. Therefore $(S, \ast)$ is a semigroup.

**Definition 3.1.** The semigroup $S$ constructed in Lemma 2.1 is called an $n$-inflation of a semigroup $T$.

It is obvious that 1-inflation is the inflation, and that 2-inflation in the strong inflation. In those cases the condition (2.3) of Lemma 2.1 it not necessary.

The following theorem gives a characterization of an $n$-inflation of semigroups, which shows that here we have the case of retract extensions.

**Theorem 2.1.** A semigroup $S$ is an $n$-inflation of a semigroup $T$ if and only if $S^{n+1} \subset T$ and $S$ is a retract extension of $T$.

**Proof.** Let $S$ be an $n$-inflation of a semigroup $T$. Then by (2.2) $T$ is an ideal of $S$. Assume $u \in S^{n+1}$, i.e. $u = s_1 \ast s_2 \ast \cdots \ast s_{n+1}$, $s_r \not\in T$ ($r = 1, 2, \ldots, n + 1$). Let $s_r \in X_1^{a_r}$ where $a_r \in T$. Then

$$u = s_1 \ast s_2 \ast \cdots \ast s_{n+1} = \Phi^{(a_1, a_2)}_{(1, 1)}(s_1, s_2) \ast s_3 \ast \cdots \ast s_{n+1}$$

If $2 > n$, then $\Phi^{(a_1, a_2)}_{(1, 1)}(s_1, s_2) = u_1 \in T$, so $u \in T$.

If $2 \leq n$, then

$$u = u_1 \ast s_3 \ast \cdots \ast s_{n+1}, \quad u_1 \in X_{t_1}^{a_1, a_2}, \quad 2 \leq t \leq n$$

$$= \Phi^{(a_1, a_2, a_3)}_{(1, 1)}(u_1, s_3) \ast s_4 \ast \cdots \ast s_{n+1}$$

If $t_1 + 1 > n$, then $\Phi^{(a_1, a_2, a_3)}_{(1, 1)}(u_1, s_3) = u_2 \in T$, so $u \in T$. 

**Theorem 2.2.** A semigroup $S$ is an $n$-inflation of a semigroup $T$ if and only if $S^{n+1} \subset T$ and $S$ is a retract extension of $T$.
If \( t_1 + 1 \leq n \) then \( u = u_2 * s_3 * \cdots * s_{n+1}, \quad u_2 \in X_{t_2}^{a_2 a_3}, \quad 3 \leq t_2 \leq n. \)

Continuing this procedure we have that: if \( t_{n-2} + 1 \leq n \), then \( \Phi_{\{s_1, \ldots, s_{n-1}, a_n\}} \).

\( (u_{n-2}, s_n) = u_{n-1} \in T \), so \( u \in T \), and if \( t_{n-2} + 1 \leq n \), then \( u = \Phi_{\{s_1, \ldots, s_{n+1}\}} \).

\( (u_{n-1}, s_{n+1}) \in T \), (since \( n + 1 > n \)).

In other cases \((r \in X_{h_r}^a, 1 < k_r \leq n)\) we have also that \( u \in T \). Thus \( S^{n+1} \subset T \).

Define a mapping \( \Phi : S = \bigcup_{a \in T} Y_a \rightarrow T \) by \( \Phi(Y_a) = a \). For any \( x, y \in S \) there exist \( a, b \in T \) such that \( x \in Y_a, y \in Y_b \), i.e. \( x \in X_i^a, y \in X_j^b \), for some \( 1 \leq i, j \leq n \). So

\[ \Phi(x \ast y) = \Phi(\Phi_{(i, j)}^{(a, b)}(x, y)), \quad \Phi_{(i, j)}^{(a, b)}(x, y) \in X_k^a \subset Y_k \]

for some \( i + j \leq k \leq n \) if \( i + j \leq n \), and \( \Phi(x \ast y) = ab \) if \( i + j > n \). Now by the definition of \( \Phi \) we have \( \Phi(x \ast y) = ab = \Phi(x) \Phi(y) \). It is clear that \( \Phi(x) = x \) for all \( x \in T \). Therefore, \( S \) is a retract extension of \( T \).

Conversely, let \( n \) be the smallest positive integer such that \( S^{n+1} \subset T \) and let \( \Phi \) be a retraction of \( S \) onto \( T \). An arbitrary \( a \in T \) is in one of the following sets \( S \setminus S^2, S^2 \setminus S^3, \ldots, S^{n-1} \setminus S^n, S^n. \) For \( a \in S^{n-r} \setminus S^{n-r+1} \) for some \( 0 \leq r \leq n - 1 \) we define the sets: \( Y_a = \Phi^r(a), \)

\[ X_i^a = Y_a \cap (S \setminus S^2) \]

\[ X_2^a = Y_a \cap (S^2 \setminus S^3) \]

\[ \vdots \]

\[ X_{n-r+1}^a = Y_a \cap (S^{n-r-1} \setminus S^{n-r}) \]

\[ X_{n-r}^a = Y_a \cap S^{n-r} \]

\[ X_{n-r+2}^a = \cdots = X_n^a = \emptyset. \]

It is clear that the conditions (2.1) hold for every \( X_i^a \) and \( X_j^b \) \((1 \leq i, j \leq n)\).

If \( a \in T \), then \( Y_a = \bigcup_{i=1}^n X_i^a \) and so \( S = \bigcup_{a \in T} Y_a \). For \( x, y \in S \) there exist \( a, b \in T \) such that \( x \in Y_a, y \in Y_b \). So by Proposition 1.1. we have that

\[ (2.4) \]

\[ Y_a Y_b \subset Y_{ab} \]

Let \( x \in X_i^a, y \in X_j^b, a \in S^{n-r} \setminus S^{n-r+1}, b \in S^{n-p} \setminus S^{n-p+1} \) where \( 0 \leq r, p \leq n - 1 \). Then

\[ x \in X_i^a = Y_a \cap (S^i \setminus S^{i+1}) \] and \( y \in Y_j^b = Y_b (S^j \setminus S^{j+1}), \quad 1 \leq i \leq n-r, \quad 1 \leq j \leq n-p. \)

Then \( xy \in S^i S^j = S^{i+j} \) and if \( i + j \leq n \) we have that \( xy \in \bigcap_{p=i+1}^n X_p^{ab} \). If \( i + j > n \), then \( xy = ab \in T \). For \( x \in X_i^a, b \in T \) we have that \( xb = ab, \ bx = ba \). In this way functions \( \Phi^{(a, b)}_{(i, j)} \) from Lemma 2.1. are defined and the condition (2.3) holds.

Definition 2.2. If the first condition (2.2.) in the construction of an \( n \)-inflation in replaced by: For \( 1 \leq i, j \leq n \) let there exists a \( k \in \{i + j, i + j + 1, \ldots, n\} \) and

\[ \Phi^{(a, b)}_{(i, j)} : X_i^a \times X_j^b \rightarrow X_k^a \]
then the semigroup \((S, \ast)\) is called the strong \(n\)-inflation of \(T\).

The following theorem is proved similarly as the previous one.

**Theorem 2.2.** A semigroup \(S\) is a strong \(n\)-inflation of a semigroup \(T\) if and only if \(S\) is an \(n\)-inflation of \(T\) and the relation determined by the following partition \(\{S \setminus S^2, S^2 \setminus S^3, \ldots, S^{n-1} \setminus S^n, S^n\}\) is a congruence of \(S\).

**Example 1.** The semigroup \(S\) given by the table 1 is a 4-inflation of \(T = \{a, b\}\). Here we have \(X_a^b = \{d, g\}, \ X_a^e = \{f\}, \ X_d^e = \{e\}, \ X_e^e = \{a, c\}, \ X_1^b = X_2^b = \emptyset, \ X_3^b = \{b\}. \) \(S\) is not strong 4-inflation of \(T\). Since \(d \cdot d = a \in X_a^a\) and \(g \cdot g = f \in X_a^a\).

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**Example 2.** The semigroup \(S\) gives by the table 2 is a strong 3-inflation of \(T = \{0\}\). Here we have \(X_1^0 = \{c, d\}, \ X_2^0 = \{b\}, \ X_3^0 = \{0, a\}.

In particular, if \(T = \{0\}\) then nilpotent semigroups of nilpotency class \(\leq n + 1\) are described by the following theorem which is directly proved by means of Theorem 2.1.

**Theorem 2.3.** Let \(X_i, \ i = 1, 2, \ldots, n\) be sets, let \(0\) be a fixed element such that \(0 \in X_n, \ X_i \cap X_j = \emptyset\) if \(i \neq j\), and let

\[
\Phi_{(i,j)} : X_i \times X_j \to \bigcup_{v=i+j}^n X_v \quad \text{if } i + j \leq n, \quad \Phi_{(i,j)}(x, y) = 0 \quad \text{if } i + j > n
\]

be functions such that

\[
(\forall s \geq i + j)(\forall s \geq j + k)\Phi_{(i,s,k)} (\Phi_{(i,j)}(x, y), z) = \Phi_{(i,t)} (x\Phi_{(j,k)}(y, z))
\]

where \(i + j \leq n\) or \(j + k \leq n\) or \(i + t \leq n\) or \(s + k \leq n\). Define a multiplication \(\ast\) on \(S = \bigcup_{v=1}^n X_v\) by:

\[
x \ast y = \Phi_{(i,j)}(x, y) \quad \text{if} \quad x \in X_i, \ y \in X_j, \ 1 \leq i, j \leq n.
\]

Then \((S, \ast)\) is a semigroup and \(S^{n+1} = 0\) and conversely, every nilpotent semigroup of nilpotency class \(\leq n + 1\) can be so constructed.
3. n-inflation of a union of groups

In the preceding section we considered \( n \)-inflations of a semigroup \( T \) in the general case. In this section we give characterization for those cases when \( T \) is a union of groups, a semilattice of groups, and so on.

**Theorem 3.1.** The following conditions are equivalent on a semigroup \( S \):

(i) \( S \) is an \( n \)-inflation of a union of groups;

(ii) \( (\forall x, y \in S) x S^{n-1} y = x^2 S^n y^2 \);

(iii) \( S^{n+1} \) is a union of groups and

\[
(x_1, \ldots, x_{n+1} \in S) (x_i^{n+1} \in G_{e_i} \Rightarrow x_1 \cdots x_{n+1} = e_1 x_1 x_2 \cdots x_{n+1} e_{n+1}).
\]

**Proof.** (i)⇒(ii). Let \( S \) be an \( n \)-inflation of a union of groups \( T \). Then \( S^{n+1} = T \) is an ideal of \( S \) and there exists a retraction \( \varphi : S \to S^{n+1} \) (Theorem 2.1.1). For any \( x, x^2, x^3, \ldots, x_n, y \in S \) there exists \( e, f \in E(S) \) such that \( \varphi(x) \in G_e \) and \( \varphi(y) \in G_f \), so

\[
xx_2 x_3 \ldots x_n y = \varphi(x) \varphi(x_2) \varphi(x_3) \ldots y \varphi(x_n) \varphi(y)
\]

\[
= \varphi(x^{n+1}) \varphi(x^{-1}) \varphi(x_2) \ldots y \varphi(x_n) \varphi(y^{-n}) \varphi(y^{n+1})
\]

\[
\in x^{n+1} S^n y^{n+1} \subset x^2 S^n y^2.
\]

Thus \( x S^{n-1} y \subset x^2 S^n y^2 \subset x S^{n-1} y \) and therefore (ii) holds.

(ii)⇒(iii). Let \( x, y \in S \). Then

\[
x S^{n-1} y = x^2 S^n y^2 = (x^{n+1})^2 S^n (y^{n+1})^2
\]

so \( x^{n+1} \in x S^{n-1} x = (x^{n+1})^2 S^n (x^{n+1})^2 \), i.e. \( x^{n+1} \) is completely regular (Lemma I, 5.1.3). So \( x^{n+1} \in G_e \) for some \( e \in E(S) \). Let \( u \in S^{n+1} \). Then

\[
u = s_1 s_2 \ldots s_{n+1} \in s_1 S^{n-1} s_{n+1} = s_1^{n+1} S^n s_{n+1} = e_1 s_1^{n+1} S^n s_{n+1} e_{n+1}
\]

where \( s_1, s_{n+1} \in G_{e_1}, s_{n+1} \in G_{e_{n+1}} \), and \( e_1, e_{n+1} \in E(S) \). Thus \( u = e_1 u = u e_{n+1} \). This proves that the second condition of (iii) is fulfilled. Now

\[
u = e_1 u = e_1 e_1 \ldots e_1 u \in e_1 S^{n-1} u = e_1 S^n u^2 \in S u^2
\]

and similarly \( u \in u^2 S \). So \( u \in u^2 S u^2 \), i.e. \( S^{n+1} \) is a union of groups (Lemma I 5.1.3).

(iii)⇒(i). Since \( S^{n+1} \) is a union of groups we have that every regular element from \( S \) is completely regular, i.e. \( S \) is a GV-semigroup. Now by Theorem X.1.1.3 (see also [15]) we have that \( S \) is a semilattice \( Y \) of semigroups \( S_{\alpha} \), where \( S_{\alpha} \) is a nil-extension of a completely simple semigroup \( P_{\alpha} (\alpha \in Y) \). It is clear that \( S_{\alpha}^{n+1} = P_{\alpha} \). Define a mapping \( \varphi : S = \bigcup_{\alpha \in Y} S_{\alpha} \to T = \bigcup_{\alpha \in Y} P_{\alpha} \) by

\[
\varphi = \varphi \mid S_{\alpha} : S_{\alpha} \to P_{\alpha}; \quad \varphi_{\alpha}(x_{\alpha}) = x_{\alpha} = x_{\alpha} e_{\alpha}, \text{ if } x_{\alpha}^{n+1} \in G_{e_{\alpha}}.
\]
Then \( \varphi \) maps \( S_\alpha \) onto \( P_\alpha \) and \( \varphi(x_\alpha) = x_\alpha \) for \( x_\alpha \in P_\alpha \). Furthermore

\[
\varphi(x_\alpha)\varphi(y_\beta) = x_\alpha e_\alpha y_\beta e_\beta = e_\alpha x_\alpha y_\beta e_\beta \quad \text{(by Theorem I.4.3, [3])}
\]

\[
= e_\alpha e_\alpha \ldots e_\alpha x_\alpha y_\beta \quad \text{(by the hypothesis)}
\]

\[
= e_\alpha e_\alpha \ldots e_\alpha x_\alpha y_\beta e_\alpha \beta\quad \text{(since \( S \) is a semilattice \( Y \) and}
\]

\[
= e_\alpha x_\alpha y_\beta e_\alpha \beta e_\alpha \beta \quad \text{(by the hypothesis)}
\]

\[
= x_\alpha y_\beta e_\alpha \beta \quad \text{(by the hypothesis)}
\]

\[
= \varphi(x_\alpha y_\beta)
\]

for all \( x_\alpha \in S_\alpha, y_\beta \in S_\beta \). Thus \( S \) is an \( n \)-inflation of a semigroup \( \bigcup_{\alpha \in Y} P_\alpha \), and \( S_\alpha \) is an \( n \)-inflation of \( P_\alpha \).

**Corollary 3.1.** A semigroup \( S \) is an \( n \)-inflation of a completely simple semigroup if and only if \( S^{n+1} \) is completely simple and the second condition of (i) of Theorem 3.1 holds.

**Proof.** By the proof of Theorem 3.1.

A subset \( B \) of a semigroup \( S \) is two-sided \((m,n)\)-pure if \( \exists \subseteq x_1 \ldots x_m, y_1 \ldots y_n \)

\[
=x_1 \ldots x_m y_1 \ldots y_n \quad \text{holds for every} \quad x_1, \ldots, x_m, y_1, \ldots, y_n \in S.
\]

A semigroup \( S \) is two-sided \((m,n)\)-pure if every \( b \)-ideal of \( S \) is a two-sided pure subset of \( S \), [5].

**Lemma 3.1.** Let \( S \) be a semigroup. If \( S^{n+1} \) is a semilattice of groups, then the idempotent elements of \( S \) are central.

**Proof.** By the hypothesis we have that \( S \) is two-sided \((n-k,k)\)-pure, \( 1 \leq k \leq n-1, n \geq 2 \) [5, Theorem 1]. So \( eSe (e \in E(S)) \) is a two-sided \((n-k,k)\)-pure bi-ideal of \( S \). From this it follows that

\[
x e \subseteq T e \quad \text{for every} \quad x \in S.
\]

Thus \( x e = e a e \) for some \( a \in S \) and similarly \( e x = e b e \) for some \( b \in S \). Now we have that

\[
x e = e a e = (a e) a e = e (a e) = (e x) e = (e b e) e = e b (e c) = e b e = e x.
\]

**Theorem 3.2.** The following conditions are equivalent on a semigroup \( S \):

(i) \( S \) is an \( n \)-inflation of a semilattice of groups,

(ii) \( \forall x, y \in S, (x S^{n+1} y = y^2 S^n x) \),

(iii) \( S^{n+1} \) is a semilattice of groups.

**Proof.** (i)\( \Rightarrow \) (iii) By Theorem 3.1 we have that \( S^{n+1} \) is a union of groups and since the idempotents of \( S \) are central we have that \( S^{n+1} \) is a semilattice of groups.
(iii) Rightarrow (ii). For every \( x, y \in S \) we have that \( xS^{n-1}y = x^2S^ny^2 \subset x^2S^{n-1}y' \) \([5, \text{Theorem 1}]\), i.e. \( xS^{n-1}y = x^2S^{n-1}y^2 = x^2S^ny \). Thus

\[
xS^{n-1}y = x^{n+1}S^{n+1}y = (x^{n+1})^{-1}(x^{n+1})^2S^n(y^{n+1})^2(y^{n+1})^{-1},
\]

since \( x^{n+1} \in G_e, y^{n+1} \in G_f \) for some \( e, f \in E(S) \). By Lemma 3.1 we have that the idempotents of \( S \) are central, so

\[
xS^{n-1}y = y^{n+1}(y^{n+1})^{-1}x^{n+1}S^ny^{n+1}(x^{n+1})^{-1}x^{n+1}
\]

whence \( xS^{n-1}y = y^2S^nx \).

(ii) \( \Rightarrow \) (iii). By the hypothesis we have that

\[
xS^{n-1}y = y^2S^nx \subset y^2S^{n-1}x = x^2S^ny^2 \subset xS^{n-1}y
\]

for every \( x, y \in S \). So the condition (ii) of Theorem 3.1 holds. From this and Theorem 3.1, we have that \( S^{n+1} \) is a union of groups. Since \( S \) is weakly commutative, so is \( S^{n+1} \). Thus \( S^{n+1} \) is a semilattice of groups \([2, \text{Theorem 1.1}]\).

(iii) \( \Rightarrow \) (i). By Lemma 3.1 the idempotents of \( S \) are central. Thus \( \varphi : S \rightarrow S^{n+1} \) defined by \( \varphi(x) = xe \) if \( x^{n+1} \in G_e \) is a retraction.

**Corollary 3.2.** A semigroup \( S \) is an \( n \)-inflation of a group \( T \) if and only if \( S^{n+1} = T \).

**Proof.** Trivial.

**Remark.** Semigroups from Theorem 3.2 are described in [5] by means of bi-ideals.

**Lemma 3.2.** \( S^{n+1} \) is a union of periodic groups if and only if

\[
(\forall x_1, x_2, \ldots, x_{n+1} \in S)(\exists m \in Z^+)x_1x_2\ldots x_{n+1} = (x_1x_2\ldots x_{n+1})^m.
\]

**Proof.** Trivial.

**Corollary 3.3.** A semigroup \( S \) is an \( n \)-inflation of a semilattice if and only if

\[
(\forall x_1, x_2, \ldots, x_{n+1} \in S)x_1x_2\ldots x_{n+1} = (x_1x_2x_3\ldots x_{n+1})^2
\]

**Proof.** Follows by Theorem 3.2 and Lemma 3.2.

**Theorem 3.3.** A semigroup \( S \) is an \( n \)-inflation of a union of periodic groups if and only if

\[
(\forall x_1, \ldots, x_{n+1} \in S)(\exists m \in Z^+)x_1\ldots x_{n+1} = x_1^{m+1}x_2\ldots x_{n+1}^{m+1}
\]

**Proof.** Let \( S \) be an \( n \)-inflation of a union of periodic groups. Then \( x_i^{n+1} \in G_{e_i} \) for every \( x_1, \ldots, x_{n+1} \in S \), whence \( x_i^m = e_i \) for some \( m \in Z^+ \), (since \( G_{e_i} \) are periodic groups). Now by Theorem 3.1, we obtain

\[
x_1x_2\ldots x_{n+1} = e_1x_1x_2\ldots x_{n+1} = e_1x_1x_2\ldots x_{n+1}e_{n+1} = x_1^{m+1}x_2\ldots x_{n+1}^{m+1}.
\]
Conversely, it is clear that $S$ is periodic. Assume $u \in S^{n+1}$. Then

$$u = x_1x_2 \ldots x_{n+1} = x_1^{m+1}x_2 \ldots x_nx_{n+1}^{m+1} = x_1^{km+1}x_2 \ldots x_nx_{n+1}^m e_{n+1}$$

where $x_1^{km} \in G_{e_1}, \ x_2^{km} \in G_{e_{n+1}} (k \in Z^+), \text{ since } S \text{ is periodic. Hence, } u = e_1x = ye_{n+1} \text{ for some } x, y \in S. \text{ So}

$$u = e_1u = e_1 \ldots e_1u = e_1 \ldots e_1u^{m+1} = u^{m+1}.$$ 

Now by Lemma 3.2 we have that $S^{n+1}$ is a union of periodic groups. Since $u = e_1ue_{n+1}$, and $x_i^{n+1} \in G_{e_1}$, for every $x_i^{n+1} \in S^{n+1}$ we have by Theorem 3.1 that the assertion of the theorem holds.

**Corollary 3.3.** A semigroup $S$ is an $n$-inflation of a semilattice of periodic groups if and only if

$$(\forall x_1, \ldots, x_{n+1} \in S)(\exists m \in Z^+)x_1 \ldots x_{n+1} = x_{n+1}^{m+1}x_2 \ldots x_nx_{n+1}^m.$$

**Proof** Follows by Theorem 3.2. and 3.3.

Following Nordahl, [8], we say that $S$ is an $E-m$ semigroup if the identity

$$(xy)^m = x^my^m \ (m \geq 2) \text{ holds in } S.$$

**Theorem 3.4.** The following conditions are equivalent on a semigroup $S$:

(i) $S$ is an $n$-inflation of a band;

(ii) $S^{n+1}$ is a band and $S$ is an $E-(n+1)$ semigroup;

(iii) $S$ is a band $Y$ of nilpotent semigroups $S_n$ of nilpotency class $\leq n$ and $Y \simeq E(S) = S^{n+1}$;

(iv) $S^{n+1}$ is a $E-(n+1)$ semigroups.

**Proof** (i)$\Rightarrow$(ii). Let $S$ be an $n$-inflation of a band $T$. Then by Theorem 2.1 $S^{n+1} \subseteq T$, $T$ is an ideal of $S$ and there is a retraction $\varphi : S \to T$. It is clear that $S^{n+1} = T$. Then for every $x, y \in S$,

$$(xy)^{n+1} = \varphi((xy)^{n+1}) = (\varphi(x)\varphi(y))^{n+1} = \varphi(x^ny) = \varphi(x)^{n+1}\varphi(y)^{n+1} = \varphi(x^{n+1})\varphi(y^{n+1}) = x^{n+1}y^{n+1}.$$

Thus, $S$ is an $E-(n+1)$ semigroups.

(ii)$\Rightarrow$(i). Clearly $\varphi(x) = x^{n+1}$ is a retraction from $S$ onto $S^{n+1}$.

(ii)$\Rightarrow$(iii). Since $\varphi(x) = x^{n+1}$ is a homomorphism from $S$ onto the band $S^{n+1}$ we have that ker$\varphi$ is a congruence $S$. Since $x(\ker \varphi) = \bar{x}$ for every $x \in S$ we have that ker$\varphi$ is a band kongruenz an the classes mod $\ker \varphi$ are nilpotent semigroups of nilpotency class $\leq n$. Clearly $Y \simeq E(S) = S^{n+1}$.

(iii)$\Rightarrow$(ii). This implication follows immediately.

(i)$\Rightarrow$(iv). This equivalence follows by Theorem 3.3.
The following corollaries follow easily from the results already prove

**Corollary 3.4.** The following conditions are equivalent on a semigroup $S$: 
(i) $S$ is an $n$-inflation of a semilattice;  
(ii) $S^{n+1}$ is a semilattice;  
(iii) $(\forall x_1, \ldots, x_{n+1} \in S)x_1x_2 \ldots x_{n+1} = x_1^2x_2\ldots x_n$.

**Corollary 3.5.** A semigroup $S$ is an $n$-inflation of a rectangular band if and only if 
$(\forall x_1, \ldots, x_{n+3} \in S)x_1x_2 \ldots x_{n+3} = x_1x_3x_4\ldots x_{n+1}x_{n+3}$

**References**


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