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INFLATION OF SEMIGROUPS

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Abstract. We introduce the concept of an n-inflation of a semigroup. In particular, for $n = 1$ we obtain the inflation introduced by Clifford [6], and for $n = 2$ the strong inflation introduced by Petrich [10]. We also characterize n-inflations of unions of groups, of semilattices of groups of unions of periodic groups, etc. In addition, we describe nilpotent semigroups of arbitrary nilpotency class.

1. Introduction and preliminaries

Let S and T be two disjoint semigroups and suppose that T has a zero element. A semigroup V is said to be an (ideal) *extension* of S by T if it contains S as an ideal and the Rees factor semigroup V/S is isomorphic to T. If, in addition, there is a partial homomorphism $\varphi : T \setminus 0 \to S$ such that for all $A, B \in T \setminus 0$ and $c, d \in S$:

$$A \circ B = \begin{cases} AB, & \text{if } AB \neq 0 \text{ in } T \\ \varphi(A)\varphi(B), & \text{if } AB = 0 \text{ in } T \end{cases}$$
$$A \circ c = \varphi(A)c, \quad c \circ A = c\varphi(A), \qquad c \circ d = rd$$

we say that extension V is determined by that partial homomorphism, [6].

Let V be an extension of S. Than V is a *retract extension* if there exists a homomorphism φ of V onto S and $\varphi(x) = x$ for all $x \in S$. In this case we call φ a *retraction*. Petrich [9] proved that an *extension V of a semigroup S by a semigroup T with zero is determined by a partial homomorphism if and only if V is a retract extension of S*. Here we give one more characterization of the retract extension.

Proposition 1.1. *Let T be a semigroup. With each $a \in T$ associate a set Y_a such that*

(1.1) $$a \in Y_a, \quad Y_a \cap Y_b = \emptyset \text{ if } a \neq b.$$

Let

(1.2) $$\varphi^{(a,b)} : Y_a \times Y_b \to Y_{ab}$$

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$\varphi^{(a,b)}(x,b) = \varphi^{(a,b)}(a,y) = ab$ *for all* $x \in Y_a$ *and* $y \in Y_b$ *be functions for which*

$$\varphi^{(ab,c)}(\varphi^{(a,b)}(x,y),z) = \varphi^{(a,bc)}(x,\varphi^{(b,c)}(y,z)) \tag{1.3}$$

and define a multiplication $*$ *on* $S = \bigcup_{a\in T} Y_a$ *by:*

$$x * y = \varphi^{(a,b)}(x,y) \qquad \text{if } x \in Y_a,\, y \in Y_b.$$

Then $(S,*)$ *is a semigroup and* S *is a retract extension of* T*. Conversely, every retract extension* S *of a semigroup* T *can be so constructed.*

Proof. Suppose that S fulfills the conditions of the proposition. Let $x \in Y_a$, $y \in Y_b$, $z \in T_c$. Then by (1.3) we have

$$\begin{aligned}(x * y) * z &= \varphi^{(a,b)}(x,y) * z = \varphi^{(ab,c)}(\varphi^{(a,b)}(x,y),z)\\ &= \varphi^{(a,ba)}(x,\varphi^{(b,c)}(y,z)) = x * \varphi^{(b,c)}(y,z)\\ &= x * (y * z).\end{aligned}$$

Hence $(S,*)$ is a semigroup. Define a mapping $\varphi : S \to T$ by $\varphi(Y_a) = a$. It is clear that φ is onto and that $\varphi(a) = a$ for $a \in T$. Furthermore, for $x \in Y_a$, $y \in Y_b$ we have

$$\varphi(x * y) = \varphi(\varphi^{(a,b)}(x,y)) = ab = \varphi(x)\varphi(y).$$

Thus φ is a homomorphism and by (1.2) T is an ideal of S. Therefore, S is a retract extension of T.

Conversely, let S be a retract extension of T. Then there is a homomorphism φ of S onto T such that $\varphi(a) = a$ for all $a \in T$. For $a \in T$ assume that $Y_a = \varphi^{-1}(a)$. Then $S = \bigcup_{a\in T} Y_a$ and for the sets Y_a $(a \in T)$ the condition (1.1) is satisfied.

For any $x, y \in S$ there exist $a, b \in T$ such that $x \in Y_a$, $y \in Y_b$, so that $\varphi(x) = a$, $\varphi(y) = b$. From this it follows that

$$\varphi(xy) = \varphi(x)\varphi(y) = ab \in Y_{ab}$$

i.e. $xy \in Y_{ab}$. Hence there exist the functions

$$\varphi^{(a,b)} : Y_a \times Y_b \to Y_{ab}$$

and it is clear that for these functions (1.3) holds. Since T is an ideal of S we have (1.2).

Clifford [6, p. 98] gave a construction for a special retract extension of a semigroup, the so-called inflation of a semigroup. A semigroup S is an *inflation* of a semigroup T if T is a subsemigroupof S and there is a mapping φ of S onto T such that $\varphi(x) = x$ for $x \in T$ and $xy = \varphi(x)\varphi(y)$ for $x, y \in S$. For further results concerning inflation of a semigroup, see [1], [3], [13], [14].

Petrich [10], [11], generalized Clifford's result introducing the notion of strong inflation.

Let T be a semigroup. To each $a \in T$ we associate two sets X_a and Y_a having the following properties:

$$a \in X_a,\ X_a \cap X_b = Y_a \cap Y_b = \emptyset \text{ if } a \neq b; \quad X_a \cap Y_b = \emptyset\ (a, b \in T).$$

To every pair of elements $x \in Y_a, y \in Y_b$, we associate an element $\varphi^{(a,b)}(x,y) \in X_{ab}$. Now let $Z_a = X_a \cup Y_b$ and define a multiplication $*$ on $S = \bigcup_{a \in T} Z_a$ by: if $x \in Z_a, y \in Z_b$, then

$$x * y = \begin{cases} \varphi^{(a,b)}(x,y) & \text{if } x \in Y_a, y \in Y_b \\ ab & \text{otherwise.} \end{cases}$$

Then S is a retract extension of T and $S^3 \subset T$. Conversely, every retract extension S of a semigroup T such that $S^3 \subset T$ can be so constructed. Such a semigroup S is called a strong inflation of a semigroup S. In particular for $T = 0$ nilpotent semigroups of nilpotency class ≤ 3 are described, [12, p. 135]. Moreover, a semigroup S in *n-nilpotent* if $S^n = 0$ $(n \in Z^+)$.

In this paper we introduce the notion of an n-inflation of a semigroup. For $n = 1$ we obtain the inflation and for $n = 2$ we obtain the strong inflation of semigroup. In Theorem 2.1. we describe an n-inflation of an arbitrary semigroup by means ot retraction. In section 2, also, a description of a strong n-inflation is given (Theorem 2.2.) and nilpotent semigroups of arbitrary nilpotency classes. In addition, we give characterizations of n-inflations of some special semigroups: unions of groups, semilattices of groups, unions of periodic groups and so on.

For undefined notions and notations we refer to [4], [6] and [12].

2. n-inflation of a semigroup

We introduce here the notion of an n-inflation of a semigroup.

Lemma 2.1. *Let T he a semigroup. To each $a \in T$ we associate a family of sets X_i^a $(i = 1, 2, \ldots, n)$ such that $a \in X_r^a$ for some $r \in \{1, 2, \ldots, n\}$ and*

$$X_i^s \cap X_j^b = \emptyset \quad \text{if } i \neq j; X_i^a \cap X_j^b = \emptyset \quad \text{if } a \neq b. \tag{2.1}$$

Let, for nonempty sets X_i^a and X_j^b,

$$\begin{aligned} \Phi_{(i,j)}^{(a,b)} &: X_i^a \times X_j^b \to \bigcup_{\nu = i+j} X_\nu^{ab} \quad \text{if } i + j \leq n \\ \Phi_{(i,j)}^{(a,b)}&(x,y) = ab \quad \text{if } i + j > n \\ \Phi_{(i,j)}^{(a,b)}&(a,y) = \Phi_{(i,j)}^{(a,b)}(x,y) = ab \end{aligned} \tag{2.2}$$

be functions for which:

$$(\forall s \geq i+j)(\forall t \geq j+k)\Phi_{(s,k)}^{(ab,c)}\left(\Phi_{(i,j)}^{(a,b)}(x,y), z\right) = \Phi_{(i,t)}^{(a,bc)}\left(x, \Phi_{(j,k)}^{(b,c)}(y,z)\right) \tag{2.3}$$

for all $a, b, c \in T$*, where* $i+j \leq n$ *or* $j+k \leq n$ *or* $i+t \leq n$ *or* $s+k \leq n$*.*

Let $Y_a = \bigcup_{i=1}^{n} X_i^a$ *and define a multiplication* $*$ *on* $S = \bigcup_{a \in T} Y_a$ *by: for* $x \in Y_a, y \in Y_b$*,*

$$x * y = \Phi_{(i,j)}^{(a,b)}(x, y) \quad \textit{if } x \in X_i, y \in X_b, 1 \leq i, j \leq n$$

Then $(S, *)$ *is a semigroup.*

Proof. Let $x, y, z \in S$. Then there exist $a, b, c \in T$ such that $x \in Y_a$, $y \in Y_b$, $z \in Y_c$ i.e. $x \in X_i^a$, $y \in X_j^b, z \in X_k^c$ for some $1 \leq i, j, k \leq n$. Assume that $i+j \leq n$ and $j+k \leq n$. Then

$$\begin{aligned}
(x * y) * z &= \Phi_{(i,j)}^{(a,b)}(x, y) * z, & \Phi_{(i,j)}^{(a,b)}(x, y) \in X_s^{ab}, \quad i+j \leq s \leq n \\
&= \Phi_{(s,k)}^{(ab,c)}\left(\Phi_{(i,j)}^{(a,b)}(x, y), z\right) & \\
(x * y) * z &= x * \Phi_{(j,k)}^{(b,c)}(y, z), & \Phi_{(j,k)}^{(b,c)}(y, z) \in X_t^{bc}, \quad j+k \leq t \leq n \\
&= \Phi_{(i,t)}^{(a,bc)}\left(x \Phi_{(j,k)}^{(b,c)}(y, z)\right) &
\end{aligned}$$

and by (2.3) we have associativity. In other cases it can be, in a similar way, proved that the associativity holds. Therefore $(S, *)$ is a semigroup.

Definition 3.1. The semigroup S constructed in Lemma 2.1. is called an n-inflation of a semigroup T.

It is obvious that 1-inflation is the inflation, and that 2-inflation in the strong inflation. In those cases the condition (2.3) of Lemma 2.1 it not necessary.

The following theorem gives a characterization of an n-inflation of semigroups, which shows that here we have the case of retract extensions.

THEOREM 2.1. *A semigroup* S *is an* n*-inflation of a semigroup* T *if and only if* $S^{n+1} \subset T$ *and* S *is a retract extension of* T*.*

Proof. Let S be an n-inflation of a semigroup T. Then by (2.2) T is an ideal of S. Assume $u \in S^{n+1}$, i.e. $u = s_1 * s_2 * \cdots * s_{n+1}$, $s_r \neq T$ $(r = 1, 2, \ldots, n+1)$. Let $s_r \in X_1^{a_r}$ where $a_r \in T$. Then

$$u = s_1 * s_2 * \cdots * s_{n+1} = \Phi_{(1,1)}^{(a_1,a_2)}(s_1, s_2) * s_3 * \cdots * s_{n+1}$$

If $2 > n$, then $\Phi_{(1,1)}^{(a_1,a_2)}(s_1, s_2) = u_1 \in T$, so $u \in T$.

If $2 \leq n$, then

$$\begin{aligned}
u &= u_1 * s_3 * \cdots * s_{n+1}, \quad u_1 \in X_{t_1}^{a_1 a_2}, \quad 2 \leq t \leq n. \\
&= \Phi_{(1,1)}^{(a_1 a_2, a_3)}(u_1, s_3) * s_4 * \cdots * s_{n+1}
\end{aligned}$$

If $t_1 + 1 > n$, then $\Phi_{(1,1)}^{(a_1 a_2, a_3)}(u_1, s_3) = u_2 \in T$, so $u \in T$.

If $t_1+1 \le n$ then $u = u_2 * s_3 * \cdots * s_{n+1}$, $\quad u_2 \in X_{t_2}^{a_1a_2a_3}$, $3 \le t_2 \le n$.

Continuing this procedure we have that: if $t_{n-2}+1 > n$, then $\Phi_{(t_{n-2},1)}^{(a_1\dots,a_{n-1}a_n)} \cdot$ $\cdot(u_{n-2},s_n) = u_{n-1} \in T$, so $u \in T$, and if $t_{n-2}+l \le n$, then $u = \Phi_{(t_n,1)}^{(a_1\dots,a_na_{n+1})} \cdot$ $\cdot(u_{n-1},s_{n+1}) \in T$, (since $n+1>n$).

In other cases ($_r \in X_{k_r}^{a_r}$, $1 < k_r \le n$) we have also that $u \in T$. Thus $S^{n+1} \subset T$.

Define a mapping $\Phi : S = \bigcup_{a\in T} Y_a \to T$ by $\Phi(Y_a) = a$. For any $x, y \in S$ there exist $a, b \in T$ such that $x \in Y_a, y \in Y_b$, i.e. $x \in X_i^a$, $y \in X_j^b$, for some $1 \le i, j \le n$. So

$$\Phi(x*y) = \Phi\left(\Phi_{(i,j)}^{(a,b)}(x,y)\right), \qquad \Phi_{(i,j)}^{(a,b)}(x,y) \in X_k^{ab} \subset Y_{ab}$$

for some $i+j \le k \le n$ if $i+j \le n$, and $\Phi(x*y) = ab$ if $i+j > n$. Now by the definition of Φ we have $\Phi(x*y) = ab = \Phi(x)\Phi(y)$. It is clear that $\Phi(x) = x$ for all $x \in T$. Therefore, S is a retract extension of T.

Conversely, let n be the smallest positive integer such that $S^{n+1} \subset T$ and let Φ be a retraction of S onto T. An arbitrary $a \in T$ is in one of the following sets $S \setminus S^2$, $S^2 \setminus S^3, \dots, S^{n-1} \setminus S^n$, S^n. For $a \in S^{n-r} \setminus S^{n-r+1}$ for some $0 \le r \le n-1$ we define the sets: $Y_a = \Phi^{-1}(a)$,

$$\begin{gathered} X_1^a = Y_a \cap (S \setminus S^2) \\ X_2^a = Y_a \cap (S^2 \setminus S^3) \\ \vdots \\ X_{n-r-1}^a = Y_a \cap (S^{n-r-1} \setminus S^{n-r}) \\ X_{n-r}^a = Y_a \cap S^{n-}r \\ X_{n-r+1}^a = X_{n-r+2}^a = \cdots = X_n^a = \emptyset. \end{gathered}$$

It is clear that the conditions (2.1) hold for every X_i^a and X_j^b $(1 \le i, j \le n)$.

If $a \in T$, then $Y_a = \bigcup_{i=1}^n X_i^a$ and so $S = \bigcup_{a\in T} Y_a$. For $x, y \in S$ there exist $a, b \in T$ such that $x \in Y_a, y \in Y_b$. So by Proposition 1.1. we have that

$$Y_aY_b \subset Y_{ab} \tag{2.4}$$

Let $x \in X_i^a, y \in X_j^b, a \in S^{n-r} \setminus S^{n-r+1}, b \in S^{n-p} \setminus S^{n-p+1}$ where $0 \le r, p \le n-1$. Then

$x \in X_i^a = Y_a \cap (S^i \setminus S^{i+1})$ and $y \in Y_j^b = Y_b(S^j \setminus S^{j+1}$, $1 \le i \le n-r$, $1 \le j \le n-p$.

Then $xy \in S^iS^j = S^{i+1}$ and if $i+j \le n$ we have that $xy \in \cap_{\nu=i+1}^n X_\nu^{ab}$. If $i+j > n$, then $xy = ab \in T$. For $x \in X_i^a$, $b \in T$ we have that $xb = ab$, $bx = ba$. In this way functions $\Phi_{(i,j)}^{(a,b)}$ from Lemma 2.1. are defined and the condition (2.3) holds.

Definition 2.2. If the first condition (2.2.) in the construction of an n-inflation in replaced by: For $1 \le i, j \le n$ let there exists a $k \in \{i+j, i+j+1, \dots, n\}$ and

$$\Phi_{(i,j)}^{(a,b)} : X_i^a \times X_j^b \to X_k^{ab}$$

then the semigroup $(S, *)$ is called the strong n-inflation of T.

The following theorem is proved similarly as the previous one.

THEOREM 2.2. *A semigroup S is a strong n-inflation of a simigroup T if and only if S is an n-inflation of T and the relation determined by the following partition $\{S \setminus S^2, S^2 \setminus S^3, \dots, S^{n-1} \setminus S^n, S^n\}$ is a congruenre of S.*

Example 1. The semigroup S given by the table 1 is a 4-inflation of $T = \{a, b\}$. Here we have $X_1^a = \{d, g\}$, $X_2^a = \{f\}$, $X_3 = \{e\}$, $X_4^a = \{a, c\}$, $X_1^b = X_2^b = X_3^b = \emptyset$, $X_4^b = \{b\}$. S is not strong 4-inflation of T. Since $d \cdot d = a \in X_4^a$ and $g \cdot g = f \in X_2^a$.

1	a	b	c	d	e	f	g
a	a	b	a	a	a	a	a
b	b	a	b	b	b	b	b
c	a	b	a	a	a	a	a
d	a	b	a	a	a	a	a
e	a	b	a	a	a	a	c
f	a	b	a	a	a	c	e
g	a	b	a	a	c	e	f

2	0	a	b	c	d
0	0	0	0	0	0
a	0	0	0	0	0
b	0	0	0	a	a
c	0	0	a	b	b
d	0	0	a	b	b

Example 2. The semigroup S gives by the table 2 is a strong 3-inflation of $T = \{0\}$. Here we have $X_1^0 = \{c, d\}$, $X_2^0 = \{b\}$, $X_3^0 = \{0, a\}$.

In particular, if $T = \{0\}$ then nilopent semigroups of nilpotency class $\leq n+1$ are described by the following theorem which is directly proved by means of Theorem 2.1.

THEOREM 2.3. *Let X_i, $i = 1, 2, \dots, n$ be sets, let 0 be a fixcd element such that $\in X_n$, $X_i \cap X_j = \emptyset$ if $i \neq j$, and let*

$$\Phi_{(i,j)} : X_i \times X_j \to \bigcup_{v=i+j}^{n} X_v \quad \text{if } i+j \leq n, \qquad \Phi_{(i,j)}(x, y) = 0 \quad \text{if } i+j > n$$

be functions such that

$$(\forall s \geq i+j)(\forall \geq j+k)\Phi_{(s,k)}\left(\Phi_{(i,j)}(x, y), z\right) = \Phi_{(i,t)}\left(x\Phi_{(j,k)}(y, z)\right)$$

where $i+j \leq n$ or $j+k \leq n$ or $i+t \leq n$ or $s+k \leq n$. Define a multiplication $*$ on $S = \bigcup_{v=1}^{n} X_v$ by:

$$x * y = \Phi_{(i,j)}(x, y) \quad \text{if} \quad x \in X_i,\, y \in X_j,\ 1 \leq i, j \leq n.$$

Then $(S, *)$ is a semigroup and $S^{n+1} = 0$ and conversely, every nilpotent semigroup of nilpotency class $\leq n+1$ can be so constructed.

3. n-inflation of a union of groups

In the preceding section we considered n-inflations of a semigroup T in the general case. In this sections we give characterization for those cases when T is a union of groups, a semilattice of groups, and so on.

THEOREM 3.1. *The following conditions are equivalent on a semigroup S:*

(i) *S is an n-inflation of a union of groups;*

(ii) $(\forall x, y \in S) xS^{n-1}y = x^2S^ny^2$;

(iii) *S^{n+1} is a union of groups and*

$(\forall x_1, \dots, x_{n+1} \in S)(x_i^{n+1} \in G_{e_i} \Rightarrow x_1 \dots x_{n+1} = e_1x_1x_2 \dots x_{n+1}e_{n+1})$.

Proof. (i)⇒(ii). Let S be an n-inflation of a union of groups T. Then $S^{n+1} = T$ is an ideal of S and there exists a retraction $\varphi : S \to S^{n+1}$ (Theorem 2.1.). For any $x, x2, x3, \dots, x_n, y \in S$ there exists $e, f \in E(S)$ such that $\varphi(x) \in G_e$ and $\varphi(y) \in G_f$, so

$$\begin{aligned} xx_2x_3 \dots x_ny &= \varphi(x)\varphi(x_2)\varphi(x_3) \dots y\varphi(x_n)\varphi(y) \\ &= \varphi(x^{n+1})\varphi(x^{-1})\varphi(x_2) \dots y\varphi(x_n)\varphi(y^{-n})\varphi(y^{n+1}) \\ &\in x^{n+1}S^ny^{n+1} \subset x^2S^ny^2. \end{aligned}$$

Thus $xS^{n-1}y \subset x^2S^ny^2 \subset xS^{n-1}y$ and therefore (ii) holds.

(ii)⇒(iii). Let $x, y \in S$. Then

$$xS^{n-1}y = x^2S^ny^2 = (x^{n+1})^2S^n(y^{n+1})^2$$

so $x^{n+1} \in xS^{n-1}x = (x^{n+1})^2S^n(x^{n+1})^2$, i.e. x^{n+1} is completely regular (Lemma I, 5.1. [3]). So $x^{n+1} \in G_e$ for some $e \in E(S)$. Let $u \in S^{n+1}$. Then

$$u = s_1s_1 \dots_{n+1} \in s_1S^{n-1}s_{n+1} = s_1^{n+1}S^nS_{n+1}^{n+1} = e_1s_1^{n+1}S^nS_{n+1}^{n+1}e_{n+1}$$

where $s_1^{n+1} \in G_{e_1}$, $s_{n+1}^{n+1} \in G_{en+1}$, and $e_1, e_{n+1} \in E(S)$. Thus $u = e_1u = ue_{n+1}$. This proves that the second condition of (iii) is fulfilled. Now

$$u = e_1u = e_1e_1 \dots e_1u \in e_1S^{n-1}u = e_1S^nu^2 \in Sn^2$$

and similarly $u \in u^2S$. So $u \in u^2Su^2$, i.e. S^{n+1} is a union of groups (Lemma I 5.1. [3]).

(iii)⇒(i). Since S^{n+1} is a union of groups we have that every regular element from S is completely regular, i.e. S is a GV-semigroup. Now by Theorem X.1.1. [3] (see also [15]) we have that S is a semilattice Y of semigroups S_α, where S_α is a nil-extension of a completely simple semigroup $P_\alpha (\alpha \in Y)$. It is clear that $S_\alpha^{n+1} = P_\alpha$. Define a mapping $\varphi : S = \bigcup_{\alpha \in Y}, S_\alpha \to T = \bigcup_{\alpha \in Y} P_\alpha$ by

$$\varphi_\alpha = \varphi \restriction S_\alpha : S_\alpha \to P_\alpha; \qquad \varphi_\alpha(x_\alpha) = x_\alpha = x_\alpha e_\alpha, \text{ if } x_\alpha^{n+1} \in G_{e_\alpha}.$$

Then φ_α maps S_α onto P_α and $\varphi(x_\alpha) = x_\alpha$ for $x_\alpha \in P_\alpha$. Furthermore

$$\begin{aligned}
\varphi_\alpha(x_\alpha)\varphi_\beta(y_\beta) &= x_\alpha e_\alpha y_\beta e_\beta = e_\alpha x_\alpha y_\beta e_\beta && \text{(by Theorem I.4.3. [3]}\\
&= e_\alpha e_\alpha \dots e_\alpha x_\alpha y_\beta && \text{see, also [7])}\\
&= e_\alpha e_\alpha \dots e_\alpha x_\alpha y_\beta && \text{(by the hypothesis)}\\
&= e_\alpha e_\alpha \dots e_\alpha \dots x_\alpha y_\beta e_{\alpha\beta} && \text{(since } S \text{ is a semilattice } Y \text{ and}\\
&= e_\alpha x_\alpha y_\beta e_{\alpha\beta} e_{\alpha\beta} \dots e_{\alpha\beta} && \text{by the hypothesis)}\\
&= x_\alpha y_\beta e_{\alpha\beta} e_{\alpha\beta} \dots e_{\alpha\beta} && \text{(by the hypothesis)}\\
&= x_\alpha y_\beta e_{\alpha\beta}\\
&= \varphi_{\alpha\beta}(x_\alpha y_\beta)
\end{aligned}$$

for all $x_\alpha \in S_\alpha$, $y_\beta \in S_\beta$. Thus S is an n-inflation of a semigroup $\bigcup_{\alpha\in Y} P_\alpha$, and S_α is an n-inflation of P_α.

COROLLARY 3.1. *A seimigroup S is an n-inflation of a completely simple semigroup if and only if S^{n+1} is completely simple and the second condition of* (i) *of Theorem* 3.1 *holds.*

Proof. By the proof of Theorem 3.1.

A subset B of a semigroup S is two-sided (m,n) pure if $B \cap x_1 \dots x_m S_{y-1} \dots y_n = x_1 \dots x_m B y_1 \dots y_n$ holds for every $x_1, \dots, x_m, y_1, \dots, y_n \in S$. A semigroup S is two-sided (m,n)-pure if every bi-ideal of S is a two-sided pure subset of S, [5].

LEMMA 3.1. *Let S be a semigroup. If S^{n+1} is a semilattice of groups, then the idempotent elements of S are central.*

Proof. By the hypothesis we have that S is two-sided $(n-k,k)$-pure, $1 \leq k \leq n-1$, $n \geq 2$ [5, Theorem 1]. So $eSe\,(e \in E(S))$ is a two-sided $(n-k,k)$-pure bi-ideal of S. From this it follows that

$$xe \in xe \dots e \cdot eSe \cdot e \dots e = eSe \cap xe \dots eSe \dots e \subset eSe$$

for every $x \in S$. Thus $xe = eae$ for some $a \in S$ and similarly $ex = ebe$ for some $b \in S$. Now we have that

$$xe = eae = (ee)ae = e(eae) = e(xe) = (ex)e = (ebe)e = eb(ee) = ebe = ex.$$

THEOREM 3.2. *The following conditions are equivalent on a semigroup S:*

(i) *S is an n-inflation of a semilattice of groups,*

(ii) $(\forall x, y \in S)(xS^{n+1}y = y^2 S^n x)$,

(iii) *S^{n+1} is a semilattice of groups.*

Proof. (i)⇒(iii) By Theorem 3.1 we have that S^{n+1} is a union of groups and since the indempotents of S are central we have that S^{n+1} is a semilattice of groups.

(iii)$Rightarrow$(ii). For every $x, y \in S$ we have that $xS^{n-1}y = x^2S^ny^2 \subset x^2S^{n-1}y^2 \subset xS^{n-1}y$ [5, Theorem 1] i.e. $xS^{n-1}y = x^2S^{n-1}y^2 = x^2S^ny$. Thus

$$xS^{n-1}y = x^{n+1}S^my^{n+1} = (x^{n+1})^{-1}(x^{n+1})^2S^n(y^{n+1})^2(y^{n+1})^{-1},$$

since $x^{n+1} \in G_e$, $y^{n+1} \in G_f$ for some $e, f \in E(S)$. By Lemma 3.1 we have that the indempotents of S are central, so

$$xS^{n-1}y = y^{n+1}(y^{n+1})^{-1}x^{n+1}S^ny^{n+1}(x^{n+1})^{-1}x^{n+1}$$

whence $xS^{n-1}y = y^2S^nx$.

(ii)$\Rightarrow$(iii). By the hypothesis we have that

$$xS^{n-1}y = y^2S^nx \subset y^2S^{n-1}x = x^2S^ny^2 \subset xS^{n-1}y$$

for every $x, y \in S$. So the condition (ii) of Theorem 3.1 holds. From this and Theorem 3.1. we have that S^{n+1} is a union of groups. Since S is weakly commutative, so is S^{n+1}. Thus S^{n+1} is a semilattice of groups [2, Theorem,1.1].

(iii)$\Rightarrow$(i). By Lemma 3.1 the idempotents of S are central. Thus $\varphi : S \to S^{n+1}$ defined by $\varphi(x) = xe$ if $x^{n+1} \in G_e$ is a retraction.

Corollary 3.2. *A Semigroup S is an n-inflation of a group T if and only if* $S^{n+1} = T$.

Proof. Trivial.

Remark. Semigroups from Theorem 3.2 are described in [5] by means ' of bi-ideals.

Lemma 3.2. S^{n+1} *is a union of periodic groups if and only if*

$$(\forall x_1, x_2, \ldots, x_{n+1} \in S)(\exists m \in Z^+)x_1x_2\ldots x_{n+1} = (x_1x_2\ldots x_{n+1})^m.$$

Proof. Trivial.

Corollary 3.3. *A semigroup S is an n-inflation of a semilattice if and only if*

$$(\forall x_1, x_2, \ldots, x_{n+1} \in S)x_1x_2\ldots x_{n+1} = (x_{n+l}x_2x_3\ldots x_nx_1)^2$$

Proof. Follows by Theorem 3.2 and Lemma 3.2.

Theorem 3.3. *A semigroup S is an n-inflation of a union of periodic groups if and only if*

$$(\forall x_1, \ldots, x_{n+1} \in S)(\exists m \in Z^+)x_1\ldots x_{n+1} = x_1^{m+1}x_2\ldots x_nx_{n+1}^{m+1}$$

Proof. Let S be an n-inflation of a union of periodic groups. Then $x_i^{n+1} \in G_{e_1}$ for every $x_1, \ldots, x_{n+1} \in S$, whence $x_i^m = e_i$ for some $m \in Z^+$, (since G_{e_i} are periodic groups). Now by Theorem 3.1. we obtain

$$x_1x_2\ldots x_{n+1} = e_1x_1x_2\ldots x_{n+1} = e_1x_1x_2\ldots x_{n+1}e_{n+1} = x_1^{m+1}x_2\ldots x_nx_{n+1}^{m+1}.$$

Conversely, it is clear that S is periodic. Assume $u \in S^{n+1}$. Then

$$u = x_1 x_2 \dots x_{n+1} = x_1^{m+1} x_2 \dots x_n x_{n+1}^{m+1} = x_1^{km+1} x_2 \dots x_n x_{n+1}^{km+1}$$
$$= e_1 x_1^{km+1} x_2 \dots x_n x_{n+1}^{km+1} e_{n+1}$$

where $x_1^{km} \in G_{e_1}$, $x_{n+1}^{km} \in G_{e_{n+1}}$ $(k \in Z^+)$, since S is periodic. Hence, $u = e_1 x = y e_{n+1}$ for some $x, y \in S$. So

$$u = e_1 u = e e_1 \dots e_1 u = e_1 \dots e_1 u^{m+1} =^{m+1} .$$

Now by Lemma 3.2 we have that S^{n+1} is a union of periodic groups. Since $u = e_1 u e_{n+1}$, and $x_i^{n+1} \in G_{e_1}$; for every $x_i^{n+1} \in S^{n+1}$ we have by Theorem 3.1 that the assertion of the theorem holds.

Corollary 3.3. *A semigroup S is an n-inflation of a semilattice of periodic groups if and only if*

$$(\forall x_1, \dots, x_{n+1} \in S)(\exists m \in Z^+) x_1 \dots x_{n+1} = x_{n+1}^{m+1} x_2 \dots x_n x_1^{m+1}.$$

Proof. Follows by Theorem 3.2. and 3.3.

Following Nordahl, [8], we say that S is an $E - m$ semigroup if the identity $(xy)^m = x^m y^m$ $(m \geq 2)$ holds in S.

Theorem 3.4. *The following conditions are equivalent on a semigroup S:*

(i) *S is an n-inflation of a band;*

(i) *S^{n+1} is a band and S is an E-$(n+1)$ semigroup;*

(iii) *S is a band Y of nilpotent semigroups S_α of nilpotency class $\leq n$ and $Y \simeq E(S) = S^{n+1}$;*

(iv) $(\forall x_1, \dots x_{n+1} \in S) x_1 x_2 \dots x_{n+1} = x_1^2 x_2 \dots x_n x_{n+1}^2$;

Proof. (i)$\Rightarrow$(ii). Let S be an n-inflation of a band T. Then by Theorem 2.1 $S^{n+1} \subseteq T$, T is an ideal of S and there is a retraction $\varphi : S \to T$. It is clear that $S^{n+1} = T$. Then for every $x, y \in S$,

$$(xy)^{n+1} = \varphi((xy)^{n+1}) = (\varphi(x)\varphi(y))^{n+1} = \varphi(x)\varphi(y)$$
$$= \varphi(x)^{n+1}\varphi(y)^{n+1} = \varphi(x^{n+1})\varphi(y^{n+1}) = x^{n+1} y^{n+1}.$$

Thus, S is an E-$(n+1)$ semigroups.

(ii)$\Rightarrow$(i). Clearly $\varphi(x) = x^{n+1}$ is a retraction from S onto S^{n+1}.

(ii)$\Rightarrow$(iii). Since $\varphi(x) = x^{n+1}$ is a homomorphism from S onto the band S^{n+1} we have that $\ker \varphi$ is a congruence S. Since $x(\ker \varphi)x^2$ for every $x \in S$ we have that $\ker \varphi$ is a band kongruence an the classes $\mod (\ker \varphi)$ are nilpotent semigroups of nilpotency class $\leq n$. Clearly $Y \simeq E(S) = S^{n+1}$.

(iii)$\Rightarrow$(ii). This implication follows immediately.

(i)$\Rightarrow$(iv). This equivalence follows by Theorem 3.3.

The following corollaries follow easily from the results already prove

COROLLARY 3.4. *The following conditions are equivalent on a semigroup S:*

(i) *S is an n-inflation of a semilattice;*

(ii) *S^{n+1} is a semilattice;*

(iii) $(\forall x_1, \dots, x_{n+1} \in S) x_1 x_2 \dots x_{n+1} = x_{n+1}^2 x_2 \dots x_n$.

COROLLARY 3.5. *A semigroup S is an n-inflation of a rectangul band if and only if*

$$(\forall x_1, \dots, x_{n+3} \in S) x_1 x_2 \dots x_{n+3} = x_1 x_3 x_4 \dots x_{n+1} x_{n+3}$$

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