PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 41 (55), 1987, pp. 3–15

ON THE REPRESENTATION OF S5 ALGEBRAS AND THEIR AUTOMORPHISM GROUPS

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Abstract. This paper deals with the representation theory of Boolean algebras operators and their automorphism groups. Mainly S5 algebras are considered, and it is shown that these operators can be represented by relatively complete Boolean subalgebras.

1. Introduction. We shall consider the representation theory of Boolean algebras with additional closure operators and their automorphism groups. We shall study mainly S5 closure operators, and it will appear that these operators are represented by relatively complete Boolean subalgebras introduced by Koppelberg [**4**]. First we introduce some terminology and notation.

The pair $(\mathbf{B}, *)$ denotes a Boolean algebra (abbreviated by BA) $\mathbf{B} = (B, +, \cdot, ', 0, 1)$ with an additional unary operation $*$ over B. Sums and products (finite or infinite) of elements $x_i \in B$, $i \in I$, are denoted respectively $\sum_i x_i, \prod_i x_i$. Occasionally, we use expansions $(\mathbf{B}, A)$ or $(\mathbf{B}, a)_{a\in A}$, where $A \subseteq B$. If $\mathbf{B}$ is generated by the set $A \cup \{u_1, \dots, u_n\}$, then we write $\mathbf{B} = \mathbf{A}(u_1, \dots, u_n)$. In such a case we say that $\mathbf{B}$ is a finitary extension of A. A set $\{\nu_1, \dots, \nu_n\}$ of elements of B is a partition of 1 if $1 = \sum_i \nu_i$, and $\nu_i\nu_j = 0$ for $i \neq j$. If $\mathbf{B}$ is a finitary extension of A, then $\mathbf{B}$ is finitary extension of A by partition of 1. We shall call such extensions normal. All model-theoretic notions are as in [**1**].

The following proposition for arbitrary algebras, and more generally for arbitrary models, will be useful later.

Proposition 1.1. 1° *Let* $\mathbf{A}$, $\mathbf{B}$ *be algebras of the same language* L, *and let* $\varepsilon : \mathbf{A} \to \mathbf{B}$ *be an onto homomorphism. Then the map*

$$\Phi : \mathrm{Aut}\,(\mathbf{A}, \ker\varepsilon) \to \mathrm{Aut}\,\mathbf{B}$$

defined by $\Phi(g) = h$ *iff* $h\varepsilon = \varepsilon g$, *is a homomorphism.*

AMS Subject Classification (1980): Primary 03B45, 03G10

2° *If* $\mathbf{A}$, $\mathbf{B}$ *are models of a language* L, *and* ε *is a strong homomorphism* (cf. [**1**]), *i.e. for all* n*-ary relation symbols* R *of* L, *and* $a_1, \dots, a_n \in A$, $R^{\mathbf{A}}\, a_1 \dots a_n$ *iff* $R^{\mathbf{B}}\, \varepsilon a_1 \dots \varepsilon a_n$ *then 1° still holds.*

Proof. 1° Observe that h is well-defined, as for $R_\varepsilon = textker\, \varepsilon$, given g and $x, y \in A$ we have $R_\varepsilon xy$ iff $R_\varepsilon gxgy$. Further more, if ω is an n-aryfunction symbol of L, then for $b_1, \dots, b_n \in B$ there are $a_1, \dots, a_n \in A$ such that

$$h\omega^{\mathbf{B}} b_1 \dots b_n = h\omega^{\mathbf{B}} \varepsilon a_1 \dots \varepsilon a_n = h\varepsilon\omega^{\mathbf{A}} a_1 \dots a_n = \varepsilon g \omega^{\mathbf{A}} a_1 \dots a_n =$$
$$= h\omega^{\mathbf{B}} \varepsilon g a_1 \dots \varepsilon g a_n = \omega^{\mathbf{B}} h b_1 \dots h b_n.$$

Claim 2° can be proved in a similar manner. □

If $\mathbf{A} \subseteq \mathbf{B}$, by $\mathrm{Aut}\,(\mathbf{B}/A)$ we denote the group of automorphisms of $\mathbf{B}$ which fix A pointwise. Therefore $\mathrm{Aut}\,(\mathbf{B}/A)$ is the Galois group of $\mathbf{B}$ over A.

2. Closure operators. We remind the reader about the following closure operators over Boolean algebras. A Boolean algebra with a closure operator $(\mathbf{B}, *)$ is:

1° a T-algebra if it satisfies the following axioms

$$0^* = 0, \quad x \le x^*, \quad (x+y)^* = x^* + y^*,$$

2° an $S4$-algebra if $(\mathbf{B}, *)$ is a T-algebra and $*$ satisfies the axiom $x^{**} = x^*$,

3° an $S5$-algebra if $(\mathbf{B}, *)$ is an $S4$-algebra and $*$ satisfies the axiom $x^{*\circ} = x^*$, where $^\circ$ is the dual operator, i.e. $x^\circ = x'^{*}{}'$.

We have the following slight generalization of a result of Drake [**2**]:

PROPOSITION 2.1. *Let* H *be a finite set of homomorphisms of* B *into* B *so that* $\mathrm{id}_B \in H$. *Define an operator* $*$ *on* B *in the following way:*

$$x^* = \sum_{g \in H} g(x), \quad x \in B. \tag{2.1.1}$$

Then: 1° $(\mathbf{B},^*)$ *is a* T*-algebra.*

2° *If* H *is closed under the composition of maps, then* $(\mathbf{B},^*)$ *is an S4-algebra.*

3° *If* H *is subgroup of* $\mathrm{Aut}\,\mathbf{B}$ *then* $(\mathbf{B},^*)$ *is an S5-algebra.*

The proofs of these facts are straightforward, so they are omitted. We shall discuss later which operators can be represented in the form (2.1.1). For finite BA's this problem was solved by Drake. Now we remark that a refinemant can be made in the case of complete BA's. Namely, if $\mathbf{B}$ is a k-complete BA and H is a set of cardinality $\le k$ of k-complete endomorphisms, an assertion similar to Proposition 2.1 holds.

Finally, if H is a subgroup of Aut $\mathbf{B}$, then for every $x \in B$ and $g \in H$, if * is defined by (2.1.1), we have

$$g(x^*) = \sum_{h \in H} gh(x) = \sum_{h \in H} h(x) = x^*;$$

so for all $x \in B$ the following holds: $x = x^*$ iff for all $g \in H, g(x) = x$.

3. Relativly complete sets. Let $\mathbf{B}$ be BA and A a subset of B. Then the set A is upward relatively complete in bf B iff every $x \in B$ there is a largest $a \in A$ such that $a \leq x$. We denote that bound by x°, and we say that the operator $^\circ$ is inducted by A.

A subset $A \subseteq B$ is downward relatively complete in $\mathbf{B}$ iff for every $x \in B$ there is a smallest $b \in A$ such that $x \leq b$. That bound is denoted by x^*, and we say the operator * is induced by A.

Some simple properties of these notions are stated in the following propositions. The term "relatively complete" is abbreviated by "r.e.".

PROPOSITION 3.1. *Let* $\mathbf{B}$ *be a* BA, *and* A *subset of* B. *Then*

1° *If* A *is upward r.c. in* $\mathbf{B}$, *and if* $^\circ$ *is induced by* A, *then* $0^\circ = 0,\ x^\circ \leq x,\ x \leq y \to x^\circ \leq y^\circ,\ x^{\circ\circ} = x^\circ,\quad (\forall x \in A)\ x^\circ = x.$

If, in addition, A *is closed under the operation* $\cdot$, *then also* $(xy)^\circ = x^\circ y^\circ$.

2° *If* A *is downward complete, and if* * *is induced by* A, *then* $1^* = 1,\ x \leq x^*,\ x \leq y \to x^* \leq y^*,\ x^{**} = x^*,\quad (\forall x \in A)\ x^* = x.$

3° *If* A *is upward r.c., and if* $^\circ$ *is the associated operator, then the set* $A = \{x'^{\circ\prime} : x \in B\}$ *is downward r.c.. Thus still holds if the words "upward", "downward", and the sings* $^\circ$,* *are interchanged.*

PROPOSITION 3.2. *Let* $\mathbf{B}$ *be a* BA, *and let* * *and* $^\circ$ *be induced operators as in Proposition 3.1. Then*

1° *If* A *is upward r.c. in* $\mathbf{B}$, *and* A *is closed under* $\cdot$, *then* A *is closed under the operation* $+$, *and* $(\mathbf{B},^*)$ *is an* $S4$ *algebra, and* $(A, +, \cdot)$ *is a Heyting algebra,*

2° *If* A *is a r.*, *(i.e. upward and downward r.c.) Boolean subalgebra of* $\mathbf{B}$, *then* $(\mathbf{B},^*)$ *is an* $S5$ *algebra.*

If * is a closure operator of $\mathbf{B}$, then the set $A = \{x^* : x \in B\}$ is associated to $(\mathbf{B},^*)$, and we have

PROPOSITION 3.3. 1° *If* $(\mathbf{B},^*)$ *is an* $S4$ *algebra, then* $(A, +, \cdot)$ *is a Heyting algebra.* 2° *If* $(\mathbf{B},^*)$ *is an* $S5$ *algebra, then* A *is a r.c. Boolean subalgebra of* $\mathbf{B}$.

In fact, these correspondences are 1-1; namely, if an operator is induced by a r.c. set A, then the set associated with * is A, and vice versa.

Some of the above statements are well known (see e.g. [**6**]), and the proofs of the others are simple; so, they are omitted.

4. S5 algebras. In this section we shall study in more detail S5 algebras $(\mathbf{B},^*)$ in which $\mathbf{B}$ is a finitary extension of the subalgebra $A = \{x^* : x \in B\}$. First, we consider an $S5$ algebra which will appear later as a canonical example of such a kind of algebras.

Algebra $(\mathbf{A}^n, \Delta)$. Let $\mathbf{A}$ be a Boolean algebra, $n \in \omega$, and Δ is r.c. in A^n, and therefore $(\mathbf{A}^n, \Delta)$ is an $S5$ algebra. Observe that the induced operator * is defined in the following way:

$$x^* = (a, a, \ldots, a), \text{ where } x = (x_1, x_2, \ldots, x_n), \quad a = \sum_i x_i.$$

Let $\nu_1 = (1, 0, \ldots, 0), \ldots, \nu_n + (0, \ldots, 0, 1)$, and for $a \in A$ define $\overline{a} = (a, \ldots, a)$. Then we have immediately for any $x \in A^n$, $x = (x_1, \ldots, x_n)$, $x = \sum_i \overline{x}_i \nu_i$.

Therefore, $A^n = \Delta(\nu_1, \ldots, \nu_n)$ and $\nu_1, \ldots, \nu_n$ is a partition of 1, i.e. $\mathbf{A}^n$ is a normal extension of Δ. These algebras are important since all $S5$ algebras $(\mathbf{B}, A)$, where $\mathbf{B}$ is a finitary extension of A, are generated by them.

PROPOSITION 4.1 *An $S5$ algebra $(\mathbf{B}, A)$ is a finitary extension of A iff $(\mathbf{B}, A)$ is a homomorphic image of (A^n, Δ) for some $n \in \omega$.*

Proof. Suppose $\mathbf{B} = \mathbf{A}(u_1, \ldots, u_n)$. We may assume that $u_1, \ldots, u_n$ is a partition of 1. Then the map $\theta : A^n \to B$ defined by

$$\theta(a) = \sum_i a_i u_i, \ \ a = (a_1, \ldots, a_n), \ \ a \in A^n$$

is a homomorphism of $(\mathbf{A}^n, \Delta)$ onto $(\mathbf{B}, A)$. On the other hand, if $(\mathbf{B}, A)$ is a homomorphic image of $(\mathbf{A}^n, \Delta)$ under a homomorphism θ, then $\mathbf{B} = \mathbf{A}(\theta\nu_1, \ldots, \theta\nu_n)$. ∎

The following example shows that there may exist many $S5$ algebras with the same Boolean part B. In this example we shall assume the Continuum Hypothesis (CH).

Let T_α, $\alpha \in \omega_1$, be a family of almost disjoint infinite subsets of ω, i.e. $\alpha < \beta < \omega_1$ implies $|T_\alpha \cap T_\beta| < \omega$. Furthermore, let $B = \{0, a, a', b, b', c, c', 1\}$ be an eight-element BA, and for $n \in T_\alpha$ define $S_n^\alpha = \{0, a, a', 1\}$ and for $n \in T^c$ take $S_n^\alpha = \{0, 1\}$. Finally, let D be the filter of cofinite subsets of ω, and define reduced products by

$$(\overline{\mathbf{B}}, \overline{S}_\alpha) = \prod_D (\mathbf{B}, S_n^\alpha).$$

Then we have:

1° $\prod_D \mathbf{2}$ is a proper subset of S_α and S_α is a proper subset of $\overline{B}$, $2 = \{0, 1\}$;

2° $\overline{\mathbf{B}} \cong \overline{S}_\alpha \cong \prod_D \mathbf{2}$ for all $\alpha \in \omega_1$;

3° for all $\alpha, \beta \in \omega_1$, $(\mathbf{B}, S_\alpha) \cong (\mathbf{B}, \overline{S}_\beta)$;

4° if $\alpha < \beta < \omega_1$, then $\overline{S}_\alpha \cap \overline{S}_\beta = \prod_D \mathbf{2}$.

To see that claim 1° holds, define the function f by $f(i) = c,\ \ i \in \omega$. Then $f_D \in \overline{S}_\alpha$. As T_α is infinite, there is a $g \in \prod_n S_n^\alpha$ such that $\{n \in \omega : g(n) \neq 0, 1\}$ is infinite. Then $g_D \in \overline{S}_\alpha - \prod_D \mathbf{2}$.

Claims 2° and 3° hold by the saturation property of the filter of cofinite subsets of ω under CH, see [**1**].

Furthermore, suppose $\alpha < \beta < \omega_1$ and $f_D \in \overline{S}_\alpha \cap \overline{S}_\beta$. Then for some $g \in \prod_n S_n^\alpha,\ \ h \in \prod_n S_n^\beta$ we have $f_D = g_D,\ \ f_D = h_D$. Thus

$$X = n \in : g(n) = h(n),\ \ g(n),\ \ h(n) \neq 0, 1 \subseteq T_\alpha \cap T_\beta$$

i.e. X is finite, therefore $g_D,\ h_D \in \prod_D \mathbf{2}$, so $f_D \in \prod_D \mathbf{2}$. Observe that the Stone space of $\prod_D \mathbf{2}$ is $\beta\omega - \omega$ (the growth of the discrete topology on ω).

Therefore, we constructed ω_1 different $S5$ algebras on $\mathbf{B}$, but all these algebras are isomorphic.

We shall use occasionally the following assertion:

Proposition 4.2. *If* $(\mathbf{B}, A)$ *is an* $S5$ *algebra, then the induced operator* * *satisfies*

$$\left(\sum_i a_i x_i\right)^* = \sum_i a_i x_i^*,\ \ a_1, \ldots, \alpha_n \in A,\ \ x_1, \ldots, x_n \in B.$$

Proof. If * is an $S5$ operator, then for any $x, y \in B,\ \ (x^* y)^* = x^* y^*$.

Proposition 4.3. *Suppose* $\mathbf{B}$ *is a Boolean algebra, and* $A \subseteq B$ *is Boolean subalgebra of* $\mathbf{B}$. *Furthermore, let* $u_1, \ldots, u_n$ *be partition of 1 in* $\mathbf{B}$ *such that* $\mathbf{B} = \mathbf{A}(u_1, \ldots, u_n)$. *Then* A *is r.c. in* $\mathbf{B}$ *iff for each* u_1 *there is a least* $\nu_i \in A$ *such that* $u_i \leq \nu_i$.

Proof. We see that each $x \in B$ has the form $x = \sum_i x_i \nu_i$ for some $x_i \in A$. Now, if $x = \sum_i x_i u_i,\ \ x_i \in A$, define $x^* = \sum_i x_i \nu_i$. Furthermore:

1° The map $x \to x^*$ is well defined: Assume $\sum_i x_i u_i = \sum_i y_i u_i,\ \ \ x_i, y_i \in A$. Multiplying both sides of this equation by u_i, we obtain $x_i u_i = y_i u_i$, therefore $u_i \leq x_i y_i + x_i' y_i'$. As $x_i y_i + x_i' y_i' \in A$, by definition of ν_i, we have $\nu_i \leq x_i y_i + x_i' y_i'$, so $x_i y_i = y_i \nu_i$. Hence, $\sum_i x_i \nu_i = \sum_i y_i \nu_i$.

2° Let $a \in A$. Then $a\nu_i$ is the least element in A such that $a u_i \leq a \nu_i$. Indeed, assume $t \in A$. Then we have $a u_i \leq t$ iff $u_i \leq a' + t$. Since $a' + t \in A$, it follows that $u_i \leq a' + t$ iff $\nu_i \leq a' + t$, but $\nu_i \leq a' + t$ iff $a\nu_i \leq t$, so for all $t \in A$, $a u_i \leq t$ iff $a \nu_i \leq t$, and this implies claim 2°.

3°. $\sum_i x_i \nu_i$ is the least element of A which is greater than x: Let $t \in A$ be such that $x \leq t$. Then $x u_i \leq t u_i$, so, since $x u_i = x_i u_i$, we have $x_i u_i \leq t u_i$, and therefore $x_i u_i \leq t$. Then by 2°, $x_i \nu_i \leq t$; thus $\sum_i x_i \nu_i \leq t$.

If $\mathbf{B}$ is a finitary extension of $\mathbf{A}$, then we can find conditions under which $(\mathbf{B}, A) \cong (\mathbf{A}^n, \Delta)$. These conditions are described in the following proposition:

PROPOSITION 4.4. *Let* $(\mathbf{B}, A)$ *be an* $S5$ *algebra such that* $\mathbf{B} = \mathbf{A}(u_1, \dots, u_n)$, *and let* $\theta : (\mathbf{A}^n, \Delta) \to (\mathbf{B}, A)$ *be the homomorphism constructed in Proposition* 4.1. *Then the following statements, are equivalent:*

$1^\circ \qquad \theta : (\mathbf{B}, A) \cong (\mathbf{A}^n, \Delta);$

$2^\circ \qquad (\forall i \le n)\ u_i^* = 1;$

$3^\circ \qquad (\forall a \in A)(u_i a = 0 \to a = 0), \quad i = 1, \dots, n.$

Proof. $(1^\circ \to 2^\circ)$ If $\nu_I = (1, 0, \dots, 0), \dots, \nu_n = (0, \dots, 0, 1)$, then $\nu_i^* = 1$ in $(\mathbf{A}^n, \Delta)$; thus $1 = \theta(\nu_i^*) = \theta(\nu_i)^* = u_i^*$, i.e. $u_i^* = 1$.

$(2^\circ \to 3^\circ)$ If $a \in A$, then $a^* = a$; therefore, by Proposition 4.2. we have $(au_i)^* = au_i^a st = a \cdot 1 = a$. Hence, if $au_i = 0$, then $(au_i)^* = 0$, i.e. $a = 0$.

$(3^\circ \to 1^\circ)$ We show that θ is 1-1. Assume $a, b \in A^n$, and let $\theta a = \theta b$. Then $\sum_i a_i u_i = \sum_i b_i u_i$; thus, for all $i \le n\ \ a_i u_i = b_i u_i$, i.e. $(a_i' b_i + a_i b_i') u_i = 0$. Since $a_i' b_i + a_i b_i' \in A$, using condition 2°, we have $a_i' b_i + a_i b_i' = 0$ i.e. $a_i = b_i$ for all $i \le n$. Therefore $a = b$.

5. Filters over algebras. Let $(\mathbf{B}, ^*)$ be an $S5$ algebra and $\mathbf{A}$ an r.c. subalgebra induced by *, i.e. $A = \{x^* : x \in B\}$. Furthermore, assume D is a filter over A, and let F_D be the filter of $\mathbf{B}$ induced by D. Then $F_D = \{x \in B : \exists t \in D\, (t \le x)\}$. Also, we can define a congruence relation $\sim_D$ over $\mathbf{B}$ induced by $D : x \sim_D y \ \text{ iff } \ \exists t \in D\ \ tx = ty,\ \ x, y \in B$.

If x, y are elements of $\mathbf{B}$ such that $x \sim_D y$, then for some $t \in D$, $tx = ty$; so $tx^* = (tx)^* = ty^*$, i.e.

$$\forall x, y \in B\ \ x \sim_D y \to x^* \sim_D y.$$

Therefore, $\sim_D$ is a congruence relation of the algebra $(\mathbf{B}, ^*)$ too, and we can define in a natural way an $S5$ operator in the quotient algebra $\mathbf{B}_D = \mathbf{B}/F_D$ by $(x/F_D^* = x^*/F_D,\ \ x \in B$.

Koppelberg introduced in [**4**] a relation over $\mathbf{B}$ induced by point of the dual space X of A, i.e. an ultrafilter over A. This relation is rather technical, but it plays an important role in descriptions of automorphisms of the algebra $(\mathbf{B}, A)$. We found an equivalent description of this relation in therms of the operation *. We remind the reader that in [**4**] $\mathbf{B}$ is identified with the set of global sections $\Gamma(\mathcal{S})$ of a sheaf $\mathcal{S} = (S, \pi, X, \mu)$, where $S \cup_{p \in X} B_p$, $\ \pi : S \to X$ where $\pi(s) = p$ iff $s \in B_p$, and $\mu : p \to B_p$, $\ p \in X$. Therefore, each $b \in B$ may be considered as a function from X to S. Now we review the definition of the relation mentioned above:

Definition 5.1. Let $p \in X$. For $x, y \in B$, $\ x \sim y$ at p iff there is a neighborhood u of p such that for $q \in u$, $\ x(q) = 0$ iff $y(q) = 0$.

THEOREM 5.2. *For any $p \in X$, and $x, y \in B$, the following statements are equivalent:*

$$1^\circ \ x \sim y \ \ at \ p;$$

$$2^\circ \ x^* \sim_p y^*;$$

3° $\mathbf{B}_p \models \mathbf{x}^* = \mathbf{y}^*$, *where* $\mathbf{x}$ *is a name of* x, *i.e. a new constant symbol, which is interpreted in* $\mathbf{B}_p$ *with the element* x/F_p.

Proof. In the following, if $a \in$, then $\mathrm{dual}(a)$ denotes the set $\{p \in X : a \in p\}$. Then we have the following equivalencies for $x, y \in B$ and $p \in X$. $x \sim y$ at $p \leftrightarrow$ is neighborhood u of p such that for all $q \in u$,

$$\begin{aligned}
&x(q) = 0 \text{ iff } y(q) = 0\\
\leftrightarrow &(\exists c \in p)(\forall q \ni c)(x/F_q = 0 \leftrightarrow y/F_q = 0)\\
\leftrightarrow &(\exists c \in p)(\forall q \ni c)((\exists s \in q)xs = 0 \leftrightarrow (\exists t \in q)yt = 0)\\
\leftrightarrow &(\exists c \in p)(\forall q \ni c)((\exists s \in q)s \le x' \leftrightarrow (\exists t \in q)t \in y')\\
\leftrightarrow &(\exists c \in p)(\forall q \ni c)s \le x'^0 \leftrightarrow (\exists t \in q) \le y'^0)\\
\leftrightarrow &(\exists c \in p)(\forall q \ni c)(x'^0 \in q \leftrightarrow y'^0 \in q)\\
\leftrightarrow &(\exists c \in p)(\forall q \ni c)(x^* \in q \leftrightarrow y^* \in q)\\
\leftrightarrow &(\exists c \in p)(\forall q \in \mathrm{dual}(c))(q \in \mathrm{dual}(x^*) \leftrightarrow q \in \mathrm{dual}(y^*))\\
\leftrightarrow &(\exists c \in p)\mathrm{dual}(c) \cap \mathrm{dual}(x^*) = \mathrm{duala}(c) \cap \mathrm{dual}(y^*)\\
\leftrightarrow &(\exists c \in p)\mathrm{dual}(cx^*) = \mathrm{dual}(cy^*)\\
\leftrightarrow &x^* \sim_p y^*.
\end{aligned}$$

Therefore, we have at once the equivalence ($1^\circ \leftrightarrow 2^\circ$). Since $\mathbf{B}_p = \mathbf{B}/F_p$, then the equivalence ($2^\circ \leftrightarrow 3^\circ$) is obvious.

COROLLARY 5.3. *Let* $x, y \in B$, $a \in A$. *Then*

$$ax^* = ay^* \leftrightarrow (\forall p \in \mathrm{dual}(a))(x \sim y \ \ at \ \ p).$$

Proof. If $ax^* = ay^*$ and $a \in p \in X$, then by definition of the filter F_p we have $x^* \sim_p y^*$; thus, by the previous theorem, $x \sim y$ at p.

Now, suppose $\forall p \in \mathrm{dual}(a)(x \sim y$ at p. Then for $p \in \mathrm{dual}(a)$, $x \sim y$ at p, i.e. $x^* \sim_p y^*$. Therefore, for some $c \in p$ we have $x^* c = y^* c$; thus, $x^* y^* + x^{*\prime} y^{*\prime} \in p$. Therefore, $x^* y^* + x^{*\prime} y^{*\prime}$ belongs to the filter of A generated by a; hence, $x^* y^* + x^{*\prime} y^{*\prime} \ge a$ i.e. $ax^* = ay^*$.

6. Automorphisms of S5 algebras. In this section we shall study automorphisms of $S5$ algebras. Let $(\mathbf{B},^*)$ be an $S5$ algebra and $\mathbf{A}$ the corresponding r.c. subalgebra of $\mathbf{B}$. Then we have several groups of automorphisms. These groups are:

$\mathrm{Aut}\,\mathbf{B}$, the group of all automorphisms of $\mathbf{B}$,

$\mathrm{Aut}\,(\mathbf{B},^*) = \{g \in \mathrm{Aut}\,\mathbf{B} : \forall x \in B\, g(x^*) = g(x)^*\}$,

$\mathrm{Aut}\,(\mathbf{B}, A) = \{g \in \mathrm{Aut}\,\mathbf{B} : \forall x \in A\, g(x) \in A\}$,

$\mathrm{Aut}\,(\mathbf{B}/A) = \{g \in \mathrm{Aut}\,\mathbf{B} : \forall x \in A\, g(x) = x\}$, the Galois group of $\mathbf{B}$ over A.

Some simple relations between these groups are described in the following proposition.

PROPOSITION 6.1. 1° $\mathrm{Aut}\,(\mathbf{B}/A) \lhd \mathrm{Aut}\,(\mathbf{B}, A) = \mathrm{Aut}\,(\mathbf{B},^*) \subseteq \mathrm{Aut}\,\mathbf{B}$.

2° $\mathrm{Aut}\,(\mathbf{B}/A) = \mathrm{Aut}\,(\mathbf{B}, a)_{a\in A}$.

3° *$g \in \mathrm{Aut}\,(\mathbf{B}/A)$ iff $g \in \mathrm{Aut}\,(\mathbf{B}, A)$ and $g \circ * = *$.*

Proof. 1° First we prove that $\mathrm{Aut}\,(\mathbf{B}, A)$ is a normal subgroup of $\mathrm{Aut}\,(\mathbf{B}, A)$. Assume $\theta \in \mathrm{Aut}\,(\mathbf{B}/A)$, and let $\alpha \in \mathrm{Aut}\,(\mathbf{B}/A)$. Then for $a \in A$ we have $(\alpha^{-1}\theta\alpha)(a) = \alpha^{-1}(\theta(\alpha(a))) = \alpha^{-1}(\alpha(a)) = a$, thus $\alpha^{-}\theta\alpha \in \mathrm{Aut}\,(\mathbf{B}/A)$.

Now we prove $\mathrm{Aut}\,(\mathbf{B}, A) = \mathrm{Aut}\,(\mathbf{B},^*)$. Assume $\theta \in \mathrm{Aut}\,(\mathbf{B}, A)$, and let $a = x$, $x \in B$. Then $(\mathbf{B}, A) \models$ "a is the least z in A such that $x \le z$" so, $(\mathbf{B}, A) \models$ "θa is the least z in A such that $\theta x \le z$", i.e. $\theta(a) = \theta(x)^*$, and therefore $\theta(x^*) = \theta(x)^*$. Hence, we have proved $\theta \in \mathrm{Aut}\,(\mathbf{B},^*)$, i.e. $\mathrm{Aut}\,(\mathbf{B}, A) \subseteq \mathrm{Aut}\,(\mathbf{B},^*)$.

Now assume $\theta \in \mathrm{Aut}\,(\mathbf{B},^*)$. If $a \in A$, then $a^* = a$, so $\theta(a) = \theta(a^*) = \theta(a)^*$. i.e. $\theta(a) \in A$ iff $\theta a \in A$; so, $\theta \in \mathrm{Aut}\,(\mathbf{B}, A)$, i.e. $\mathrm{Aut}\,(\mathbf{B},^*) \subseteq \mathrm{Aut}\,(\mathbf{B}, A)$.

2° Remember that $(\mathbf{B}, a)_{a\in A}$ is a simple expansion of the model $\mathbf{B}$ by the individual constans $a \in A$.

3° Assume $\theta \in \mathrm{Aut}\,(\mathbf{B},^*)$ and $\theta \circ * = *$. Then for $a \in A$ we have $\theta(a) = \theta(a^*) = a^* = a$, i.e. $\theta \in \mathrm{Aut}\,(\mathbf{B}/A)$.

In the case of $S5$ algebras which are finitary extensions, there is a condition which admits automorphisms with special properties:

THEOREM 6.2. *Let $\mathbf{A}$ be a r.c. subalgebra of a* BA $\mathbf{B}$, *and let $\mathbf{B} = \mathbf{A}(u_1, \dots, u_n)$ $\mathbf{B} = \mathbf{A}(\nu_1, \dots, \nu_n)$ be two normal extensions of $\mathbf{A}$. Assume that for all $i \le n$ we have $u_i^* = \nu_i^*$. Then there is a $g \in \mathrm{Aut}\,(\mathbf{B}/A)$ such that $g(u_i) = \nu_i$ for all $i \le n$.*

Proof. Consider the map $g : B \to B$ defined by

$$g : x_1u_1 + \cdots + x_nu_n \mapsto x_1\nu_1 + \cdots + x_n\nu_n, \quad x_1, \dots, x_n \in A.$$

Then $g \in \mathrm{Aut}\,(\mathbf{B}/A)$ and $g(u_i) = \nu_i$. $1 \le i \le n$. To see that, we first prove:

(1) For all $i \le n$, $u_i^* = \nu_i^*$ iff for all $a \in A$, $a \le u_i' \leftrightarrow a \le \nu_i'$. This follows from the following. Assume $x \in A$; then

$$x \le u_i' \leftrightarrow x \le {u_i}'^0 \leftrightarrow u_i^* \le x' \leftrightarrow \nu_i^* \le x' \leftrightarrow x \le {\nu_i}'^0 \leftrightarrow x \le \nu_i'.$$

On the other hand if $(\forall a \in A) a \le u_i$przir $\leftrightarrow a \le \nu_i'$, then, as above $(\forall x \in A) u_i^* \le x' \leftrightarrow \nu_i^* \le x'$; so, $u_i^* = \nu_i^*$, i.e. (1) holds.

Now we check, for example, that g is well defined. Assume $x_1u_1 + \cdots + x_nu_n = y_1u_1 + \cdots + y_nu_n$. Then, as $\{u_i : i \le n\}$ is a partition of 1, we have $x_iu_i = y_iu_i$ for all $i \le n$; so, $x_iy_i' + x_i'y_i \le u_i'$; thus, $x_iy_i' + x_i'y_i \le \nu_i'$, i.e. $x_i\nu_i = y_i\nu_i$. Hence,

$$x_1\nu_1 + \cdots + x_n\nu_n = y_1\nu_1 + \cdots + y_n\nu_n.$$

It is easy to see now that $g \in \mathrm{Aut}\,(\mathbf{B}/A)$; so we omit the rest of the proof.

In the following examples we shall illustrate the last theorem. In all cases, S_n denotes the set of all permutations of the set $\{1, 2, \dots, n\}$, and $\{u_1, \dots, u_n\}$ is a partition of 1.

Example 6.3. Let $\pi \in S_n$ and consider an $S5$ algebra $(\mathbf{A}^n, \Delta)$, $A^n = \Delta(u_1, \dots, u_n)$. Then for all $i \leq n$, $u_i^* = 1$; hence, there is $g \in \mathrm{Aut}\,(\mathbf{A}^n, \Delta)$ such that $g(u_i) = u_{\pi(i)}$.

Example 6.4. Let $(\mathbf{B}, A)$, $\mathbf{B} = \mathbf{A}(u_1, \dots, u_n)$ be an $S5$ algebra, and $\pi \in S_n$. Furthermore, assume that for some $p \in X = \mathrm{dual}\, A$, for all $i \leq n$, $u_i \sim u_{\pi(i)}$ at p. Then there is a $g \in \mathrm{Aut}\,(\mathbf{B}/A)$such that for $i \leq n$, $g(u_i) \sim_p u_{\pi(i)}$.

Proof. Assume $u_i \sim u_{\pi(i)}$ at p. Then by Theorem 5.2 there is a $c_i \in p$ such that $u_i^* c_i = u_{\pi(i)}^* = c_i$. Therefore, for $c = c_1 c_2 \dots c_n$, we have $c \in p$ and $u_i^* c = u_{\pi(i)}^* c$ for all $i \leq n$. The set $\{c_i' u_i + c u_{\pi(i)} : 1 \leq i \leq n\}$ is a partition of 1, and also:

$$(c_i' u_i + c u_{\pi(i)})^* = c' u_i^* + c u_{\pi(i)}^* = c' u_i^* + c u_i^* = u_i^*.$$

By Theorem 6.2 there is $g \in \mathrm{Aut}\,(\mathbf{B}/A)$ such that $g(u)i) = c_i' u_i^* + c u_{\pi(i)}^*$. Thus, $g(u_i)/F_p = (c_i' u_i + c u_{\pi(i)})/F_p = c'/F_p \cdot u_i/F_p + c/F_p \cdot u_{\pi(i)}/F_p = u_{\pi(i)}/F_p$, as $c \in p$, $c' \neq p$. Therefore, $g(u_i) \sim_p u_{\pi)i)}$.

This example is related to Lemma 2.3 in [**4**], and we shall return to it later.

Example 6.5. Let $(\mathbf{B}, A), \mathbf{B} = A(u_1, \dots, u_n)$ be an $S5$ algebra, and assume $\{a_\pi : \pi \in S_n\}$ is a partition of 1 in A such that

$$\forall \pi \in S_n \quad a_\pi u_r^* = a_\pi u_{\pi(r)}^*.$$

Then there is a $g \in \mathrm{Aut}\,(\mathbf{B}/A)$ such that

$$g(u_r) = \sum_\pi a_\pi u_{\pi(r)}, \qquad 1 \leq r \leq n. \tag{6.5.1}$$

Proof. It is easy to see that $\left\{\sum_\pi a_\pi u_{\pi(r)} : 1 \leq r \leq n\right\}$ is a partition of 1. Furthermore:

$$\left(\sum_\pi a_\pi u_{\pi(r)}\right)^* = \sum_\pi a_\pi u_{\pi(r)}^* = \sum_\pi a_\pi u_r^* = u_r^*.$$

Therefore, by Theorem 6.2 there is a $g \in \mathrm{Aut}\,(\mathbf{B}/A)$ with the required property.

This example is a new proof of Theorem 2.4 (a) in [**4**]. We note that 6.5.1 holds for an appropriate partition a_π, $\pi \in S_n$, of 1 in A.

In the following proposition we state some properties of the Galois group of **B** over A.

PROPOSITION 6.6. *Let Xbe the dual space of A. Then:*

1° *for everi $p \in X$, every $g \in \mathrm{Aut}\,(\mathbf{B}/A)$ generates a $\overline{g} \in \mathrm{Aut}\,\mathbf{B}_p$;*

2° $\mathrm{Aut}\,(\mathbf{B}/A) = \bigcap_{p \in X} \mathrm{Aut}\,(\mathbf{B}, \sim_p)$;

3° *for all $p \in X$, $x, y \in B$ and $g \in \mathrm{Aut}\,(\mathbf{B}/A)$ we have $gx \sim_p y \to x \sim y$ at p.*

Proof. 1° Let $g \in \mathrm{Aut}\,(\mathbf{B}/A)$ and assume $x \sim_p y$ for some $x, y \in B$. Then there is a $t \in p$ such that $tx = ty$; hence

$$tg(x) = g(tx) = g(ty) = tg(y), \text{ i.e. } g(x) \sim_p g(y)$$

and $\mathrm{Aut}\,(\mathbf{B}/A) \subseteq \mathrm{Aut}\,(\mathbf{B}, \sim_p)$. Thus, by Proposition 1.1 there is a homomorphism $\Phi : \mathrm{Aut}\,(\mathbf{B}/A) \to \mathrm{Aut}\,\mathbf{B}$ defined by $\overline{g} \circ k_p = k_p \circ g$, where $\Phi : g \mapsto \overline{g}$, and $k_p \mathbf{B} \to \mathbf{B}_p$ is the natural homomorphism. Observe that $\sim_p = \ker k_p$.

2° By 1° we have $\mathrm{Aut}\,(\mathbf{B}/A) \subseteq \mathrm{Aut}\,(\mathbf{B}, \sim_p)$ for all $p \in X$. So, suppose for all $p \in X, g \in \mathrm{Aut}\,(\mathbf{B}, \sim_p)$. Let $a \in A$, and assume $g(a) \neq a$. Then there is a $p \in X$ such that $a \in p$ and $g(a)' \in p$, or $a' \in p$ and $g(a) \in p$. In the first case we have $a \sim_p 1$ and $g(a) \sim_p 0$, and in the second, $a \sim_p 0$ and $g(a) \sim_p 1$, a contradiction. Therefore for all $a \in A$, $g(a) = a$, i.e. $g \in \mathrm{Aut}\,(\mathbf{B}/A)$.

3° Assume $gx \sim_p y$. Then $g(x)t = yt$ for some $t \in p \in X$; so $g(xt)^* = (ty)^*$. As $(ty)^* \in A$, we have $g((ty)^*) = (ty)^*$, thus $g((xt)^*) = g((ty)^*)$, and so $(tx)^* = (ty)^*$ i.e. $x^* \sim_p y^*$. By Theorem 6.2 it follows that $x \sim y$ at p.

If $g \in \mathrm{Aut}\,(\mathbf{B}/A)$ and $p \in X$ dual $\mathbf{A}$, then by the last proposition, g generates a $g \in \mathrm{Aut}\,\mathbf{B}_p$. As $x \in B$ is of the form $\sum_i x_i u_i$, such that for all $i \in I$, $x_i \in p$ or $x' \in p$ or $x_i' \in p$, it follows that $\overline{g}(x/F_p) = \sum_i g(u_i)/F_p$, F_p is the filter of B generated by p. The algebra B_p is finite, and u_i/F_p are atoms of B_p; so, they are permuted by $\overline{g}$, i.e. there is a $\pi \in S_n$ such that $\overline{g}(u_i/F_p) = u_{\pi(i)}/F_p$, i.e. $g(u_i)/F_p = u_{\pi(i)}/F_p$ for all $i \leq n$. As

$$u_i^*/F_p = g(u_i^*)/F_p = (g(u_i)/F_p)^* = (u_{\pi(i)}/F_p)^* = u_{\pi(i)}^*/F_p,$$

it follows that

$$u_i^* \sim_p u_{\pi(i)}^*, \text{ i.e. } u_i \sim u_{\pi(i)} \text{ at } p \text{ for all } i \leq n.$$

This connection between points of X and elements of S_p in considered in [**4**], and there it is stated that $\pi \in S_n$ is compatible with $p \in X$ iff for all $1 \leq i \leq n$, $u_i \sim u_{\pi(i)}$ at p. Furthermore $g \in \mathrm{Aut}\,(\mathbf{B}/A)$ is said to be induced by $\pi \in S_n$ at $p \in X$ iff for all $i \leq n$, $g(u_i) \sim_p u_{\pi(i)}$ (cf. [**4**, p. 238]). Therefore, by the former and Example 6.4 we have at once Lemma 2.3 in [**4**].

LEMMA 6.7. *$\pi \in S_n$ is compatible with $p \in X$ iff there is a $g \in \mathrm{Aut}\,(\mathbf{B}/A)$ which is induced by π at p.*

As we have seen, $S5$ algebras $(\mathbf{B}, A)$ such that $\mathbf{B}$ is a finitary extension of A, are generated by algebras $(\mathbf{A}^n, \varDelta)$. Now we shall see that $\mathrm{Aut}\,(\mathbf{B}/A)$ has a similar property with respect to $\mathrm{Aut}\,(\mathbf{A}^n/\varDelta)$. Namely, we have the following theorem.

THEOREM 6.8. *If* $\mathbf{B} = A(u_i, \dots, u_n)$ *is normal extension of* A*, then* $\mathrm{Aut}\,(\mathbf{B}/A)$ *is a homomorphic image of a subgroup of* $\mathrm{Aut}\,(\mathbf{A}^n/\Delta)$.

Proof. Let $\theta : \mathrm{Aut}\,(\mathbf{A}^n/\Delta) \to (\mathbf{B}, A)$, $\theta \restriction \Delta : \Delta \cong \mathbf{A}$, be the homomorphism constructed as in Proposition 4.1. For $R_\theta = \ker\theta$ define

$$G_\theta = \mathrm{Aut}\,(\mathbf{A}^n, \Delta, R_{\theta,a})_{a\in A}.$$

Therefore, $g \in G_\theta$ iff $g \in \mathrm{Aut}\,(\mathbf{A}/\Delta)$, and for all $x, y \in A^n$, $R_\theta(x, y) \leftrightarrow R_\theta(gx, gy)$. As $G_\theta < \mathrm{Aut}\,(\mathbf{A}/\Delta)$, it suffices to show that $\mathrm{Aut}\,(\mathbf{B}/A)$ is a homomorphic image of G_θ.

By Proposition 1.1, the map $\psi : g \mapsto h$ defined by $h \circ \theta = \theta \circ g$, $g \in G_\theta$, is a homomorphism of G_θ into $\mathrm{Aut}\,(\mathbf{B}/A)$. Now we prove that ψ is onto. Let $h \in \mathrm{Aut}\,(\mathbf{B}/A)$. By the Representation Theorem 2.4 in [**4**] (see also the comment in Example 6.5), there is a partition a_π, $\pi \in S_n$, of 1 in $\mathbf{A}$ such that

$$h(u_r) = \sum a_\pi u_{\pi(r)}.$$

For $x \in A$ let $x = (x, \dots, x), \dots, \nu_n = (0, \dots, 0, 1)$. Then for all $\pi \in S_n, a_\pi \nu_r^* = a_\pi \nu_{\pi(r)}^*$; hence,by Example 6.5, there is a $g \in \mathrm{Aut}\,(\mathbf{A}/\Delta)$ such that $g(\nu_r) = \sum_{\pi\in S_n} a_{\pi(r)}$. Then we have

$$h(u_r) = \sum_\pi a_\pi u_{\pi(r)} = \sum_\pi a_\pi \theta(\nu_{\pi(r)}) = \theta\left(\sum_\pi a_\pi \nu_{\pi(r)}\right) = \theta g(\nu_r);$$

so $\psi(g) = h$, i.e. ψ is onto.

According to [**4**], groups which are obtained as bounded Boolean powers of a group H by a Boolean algebra $\mathbf{A}$, play an important part in the representation theory of $\mathrm{Aut}\,(\mathbf{B}/A)$. We recall that such groups are of the form $H(A) = \{f : X \to H\varphi$ is continuos$\}$, where H is given the discrete topology, and X is the Stone space of A. By Example 2.3 in [**4**] and Proposition 4.4 we have at once

PROPOSITION 6.9. $\mathrm{Aut}\,(\mathbf{A}^n/\Delta) \cong S_n(A)$.

By Theorem 6.8 we then have the following:

COROLLARY 6.10. *If* $(\mathbf{B}, A)$ *is an* $S5$ *algebra and* $\mathbf{B} = \mathbf{A}(u_1, \dots, u_n)$*, then* $\mathrm{Aut}\,(\mathbf{B}/A)$ *is a homomorphic image of a subgroup of* $S_n(\mathbf{A})$.

The following proposition improves a part of Proposition 2.7 in [**4**].

PROPOSITION 6.11. *Let* G *be a finite group,* $\mathbf{A}$ *a Boolean algebra and* $g_1, \dots, g_n \in G(\mathbf{A})$*. Then the subgroup* $< g_1, \dots, g_n >$ *generated by* $g_1, \dots, g_n$ *is finite, i.e. finitely generated subgroups of* $G(\mathbf{A})$ *are finite.*

Proof. If $g \in G = G(A)$, then $\tau = \{g^{-1}(\alpha) : \alpha \in \mathrm{Im}\,(g)\}$ is a finite partition of X which corresponds to g. Since g is continuous, the elements of the partition are clopen; so, there are $a_1, \ldots, a_n \in A$ such that $\tau = \{\mathrm{dual}\,(a_i) : i = 1, \ldots, k\}$. Therefore, for some $\alpha_1^j, \ldots, \alpha_{k_j}^j \in G$ we have $g_j(p) = \sum_i k_i^j(p)\alpha_i^j$, where k_i^j are the characteristic functions of $\mathrm{dual}\,(a_i^j)$, and $0 \cdot \alpha = 0,\ 1 \cdot \alpha = \alpha$. So, $g_j(p) = \alpha_i^j$ for $p \in \mathrm{dual}\,(a_i^j)$.

If $w(x_1, \ldots, x_m)$ is a group-word, then

$$w_G(g_1, \ldots, g_m) = \sum_{i_1, \ldots, i_m} k_{i_1} \ldots i_m(p) \cdot w_G(\alpha_{i_1}^1, \ldots, \alpha_{i_m}^m)$$

where $w_G(\alpha_{i_1}^1, \ldots, \alpha_{i_m}^m)$ is the value of $w_G(\alpha_{i_1}^1, \ldots, \alpha_{i_m}^m)$ in G_2, and $k_{i_1 \ldots i_m}$ is the characteristic function of $\mathrm{dual}\,(a_{i_1}^1 \cdot a_{i_2}^2 \ldots a_{i_m}^m)$. But since G is finite, there are only finitely many functions $k_{i_1 \ldots i_m}$, and values $w_G(\alpha_{i_1}^1, \ldots, a_{i_m}^m)$. Therefore there are only finitely many functions of the form $w_G(g_1, \ldots, g_m)$. Thus $< g_1, \ldots, g_m >$ is finite.

By the last theorem and Corollary 6.10 we have (cf. [**4**, Proposition 2.7]);

COROLLARY 6.12. *Every finitely generated subgroup of* $\mathrm{Aut}\,(\mathbf{B}/A)$ *is finite.*

It is easy to see that for any Boolean algebra $\mathbf{A}$, $S_2(\mathbf{A}$ is generated by characteristic functions f_a of $\mathrm{dual}\,(a)$, $a \in A$, and that all elements of $S_2(A)$ are of order 2. Thus, if $\mathbf{B} = A(u_1, u_2)$, then $\mathrm{Aut}\,(\mathbf{B}/A)$ is a sum of cyclic groups of order 2.

We shall close this section with a group-representation of the form 2.1.1 of an $S5$ operator * in $(B,^*)$ is a finitary extension of $A = \{x \in B : x^* = x\}$.

THEOREM 6.13. *If* $(\mathbf{B}, A)$ *is an* $S5$ *algebra such that* $\mathbf{B} = \mathbf{A}(u_1, \ldots, u_n)$ *and if* * *is the corresponding* $S5$ *operator, then there is a finite subgroup* $H < \mathrm{Aut}\,(\mathbf{B}/A)$ *such that for all* $x \in B$, $x^* = \sum_{g \in H} g(x)$.

Proof. By Proposition 2.7 i [**4**] there is a finite $H < \mathrm{Aut}\,(\mathbf{B}/A)$ such that for every $b \in B - A$ there is a $g \in H$ satisfying $g(b) \neq b$. If $g \in H$ and $x \in B$, then $g(x) \leq x^*$, since $x \leq x^*$ and $g(x^*) = x^*$. Thus:

$$\sum_{g \in H} g(x) \leq x^*. \tag{1}$$

Let $y = \sum_{g \in H} g(x)$. Then $x \leq y \leq x^*$, since id $\in G$. If $y < x^*$, then $y \in B - A$; so, by our assumption on H, there is an $h \in H$, such that $h(y) \neq y$. But

$$h(y) = h\left(\sum_{g \in H} g(x)\right) = \sum_{g \in H} hg(x) = \sum_{g \in H} g(x) = y,$$

a contradiction. Therefore $y = x^*$ i.e. $x^* = \sum_{g \in H} g(x)$.

In fact, y Proposition 2.7 in [**4**] we can take the H above to be cyclic.

7. Some remarks. In this section we make some remarks and note some problems concerning Boolean algebras with closure operators, whose solution might be of an interest.

7.1 There are several possible representations of $S5$ algebras. To list them, suppose $\mathbf{B}$ is a Boolean algebra. Then an $S5$ algebra over $\mathbf{B}$ can be given in the following ways:

1° $(\mathbf{B},^*)$, where * is an $S5$ operator. Then the class of such algebras is a variety.

2° $(\mathbf{B}, A)$, where $\mathbf{A}$ is a r.c. subalgebra of $\mathbf{B}$. The class of such algebras can be axiomatized by first-order axioms.

3° There is a Stone-type representation, discussed by Hansoul in [**3**].

4° $S5$ algebras can be represented by Hausdorff sheafs, as shown by Koppelberg (cf. [**4**, p. 236]).

5° If B is a finitary extension of A, then $(\mathbf{B},^*)$ can be represented by $(\mathbf{B}, H)$, where H is a certain group of automorphism of $\mathbf{B}$.

If categories with appropriate objects and morphisms are formed for each of the listed representations, it is not difficult to see that between these categories there exist natural equivalencies.

7.2. In the case of the representation 7.1.5 it is interesting to see whether the assumption that $\mathbf{B}$ is a finitary extension of $\mathbf{A}$, can be lifted (i.e., describe all $S5$ algebras for which such a representation holds).

7.3. In [**4**] it was shown thaat the first-order theory of algebras $(\mathbf{B}, A)$, where $\mathbf{A}$ is r.c. in $\mathbf{B}$, $\mathbf{B}$ is complete, and the inclusion map from A into B is complete, is decidable. For such an algebra $(\mathbf{B}, A)$, the corresponding operator * is complete (and vice versa, i.e. for any subset $S \subseteq B$, $\left(\sum_{x\in S} x\right)^* = \sum_{x\in S} x^*$). Therefore, the first-order theory of $S5$ algebras $(B,^*)$ with complete $\mathbf{B}$ and complete * is decidable. (It would be interesting to see whether the theory of all $S5$ algebras is decidable.) As a partial solution, in [**5**] it was shown that $\prod_1^0$ fragment is decidable.

7.4. It would be interesting to develop a similar analysis for other Boolean algebras with modal operators; for example, T-algebras, $S4$-algebras, or G-algebras.

7.5. Develop the model theory of $S5$ algebras.

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(Received 09 06 1986)