

**ON CERTAIN CONDITIONS WHICH REDUCE
 A FINSLER SPACE OF SCALAR CURVATURE TO
 A RIEMANNIAN SPACE OF CONSTANT CURVATURE**

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Abstract. We give certain conditions which reduce a Finsler space of scalar curvature to a Riemannian space of constant curvature.

1. Preliminaries. Let F_n be an n -dimensional Finsler space with the fundamental functional $L(x, y)$, the positive definite metric tensor $g_{ij} = 1/2\delta_i\delta_j L^2$ and the angular metric tensor $h_{ij} = g_{ij} - l_i l_j$, where $l_i = \dot{\delta}_i L$, $\dot{\delta}_i = \delta/\delta y^i$.

For a Cartan connection CT , h - and ν -covariant derivatives of a finsler tensor field X_j^i are denoted by $X_{j|k}^i$ and $X_j^i|_k$. The h -, $h\nu$ - and ν -curvature tensors of CT are R_{hjk}^i , P_{hjk}^i and S_{hjk}^i and the (ν) h -, (h) $h\nu$ - and (ν) $h\nu$ -torsion tensors of CT are R_{jk}^i , C_{jk}^i and P_{jk}^i respectively. On the otherhand H_{hjk}^i and H_{jk}^i are h -curvature tensors and (ν) h -torsion, tensors of Berwald connection $B\Gamma$ respectively.

The following relations are well known [4]:

$$(1.1) \quad H_{ijk}^h = \dot{\delta}_i H_{jk}^h$$

$$(1.2) \quad P_{ijk} = C_{ijk|o},$$

where the index o stands for transvection by y and $C_{ijk} = 1/2\dot{\delta}_k g_{ij}$

$$(1.3) \quad H_{jk}^i = H_{ojk}^i = R_{jk}^i = R_{ojk}^i,$$

$$(1.4) \quad H_{ijk}^h = R_{ijk}^h - C_{ir}^h R_{jk}^r + \{P_{ij|k}^h - P_{jr}^h P_{ki}^r - j | k\}.$$

where $j | k$ means interchange of indices j, k in the foregoing terms.

A hypersurface of F_n defined by the equation $L(x, y) = 1$, where the point $x = (x^i)$ is fixed and y^i are variables, is called indicatrix. We denote by p the projection of the tensor of the Finsler spaces on the indicatrix. For example, the

projection of the tensor T_j^i of type (1,1) of F_n on the indicatrix is $p \cdot T_j^i = h_a^i T_b^a h_j^b$, where $h_a^i = \delta_a^i - l^i l_a$, $l^i = g^{ij} l_j = L^{-1} y^i$. A tensor T satisfying $p \cdot T = T$ is called an indicatrix tensor. We have

$$(1.5) \quad \begin{array}{ll} \text{a) } p \cdot l^i = p \cdot l_i = 0, & \text{b) } p \cdot \delta_j^i = h_j^i, \\ \text{c) } p \cdot \dot{\delta}_k h_j^i = p \cdot h_j^i|_k = 0, & \text{d) } p \cdot \dot{\delta}_k h_{ij} = 2C_{ijk} \end{array}$$

2. A Finsler space of scalar curvature. A Finsler space of scalar curvature is characterized by [6] any one of the following equations:

$$(2.1a) \quad H_{jk}^i = L(Kl_j + K_j/3)h_k^i - j|k,$$

$$(2.1b) \quad \begin{aligned} H_{hj}^i &= \{l_h(Kl_j + K_j/3) + Kh_{hj} + 2K_h l_j/3 + K_{hj}/3\}h_k^i \\ &+ l^i(Kl_k + K_k/3)h_{hj} + h_h^i l_j K_k/3 - j|k \end{aligned}$$

where $K_k = L\dot{\delta}_k K$, $K_{hj} = Lp \cdot \dot{\delta}_h K_j = K_{jh}$. Specially, if the scalar K is constant, then the space is called a Finsler space of constant curvature.

PROPOSITION 2.1. A Finsler space $F_n (n \geq 3)$ of scalar curvature K satisfies

$$(2.2) \quad K_{ijk} + K_i h_{jk} - i|j = 0,$$

where $K_{ijk} = Lp \cdot \dot{\delta}_i K_{jk}$.

Proof. From (1.5) and (2.1b), we have

$$\begin{aligned} Lp \cdot \dot{\delta}_m H_{hj}^i &= (h_{hm}K_j/3 + K_m h_{hj} + 2LK C_{hjm} + 2K_h h_{jm}/3 + K_{mhj}/3)h_k^i \\ &+ h_m^i K_k h_{hj}/3 + h_h^i h_{jm} K_k/3 - j|k \end{aligned}$$

Considering the skew-symmetric part of the above equation in the indices h and m and using the fact $\dot{\delta}_m H_{hj}^i = \dot{\delta}_h H_{mjk}^i$, we get

$$[(K_m h_{hj} + 2K_h h_{jm}/3 + K_{mhj}/3)h_k^i - j|k] - h|m = 0$$

which is simplified as

$$(2.3) \quad [(K_m h_{hj} + K_{mhj})h_k^i - j|k] - h|m = 0$$

Contracting (2.3) in indices i and k , we get (2.2).

Remark 2.1. Proposition 2.1. and the definition of K_j , K_{hj} and K_{ijk} imply that when any one of them is zero, then the other two are automatically zero. $K_j = 0$ means that K is independent of y . Thus K is constant (Matsumoto [4, Prop. 26.1]). If for a Finsler space F_n of scalar curvature any one of the tensors K_i , K_{hj} and K_{ijk} vanishes, F_n is of constant curvature.

PROPOSITION 2.2. A Finsler space F_n of scalar curvature K with $P_{hij|0} = 0$ satisfies

$$(2.4) \quad h_{ih}(3KK_{jm} - K_j K_m) + h_{jh}(3KK_{im} - K_i K_m) - h|m = 0$$

Proof. A Finsler space F_n of scalar curvature K satisfies [7]

$$(2.5) \quad L^{-1}P_{hij|0} + LKC_{hij} + (K_h h_{ij} + K_i h_{hj} + K_j h_{hi})/3 = 0.$$

Since, $P_{hij|0} = 0$, (2.5) leads to

$$(2.6) \quad LKC_{hij} + (K_h h_{ij} + K_i h_{hj} + K_j h_{hi})/3 = 0.$$

Differentiating the equation above partially with respect to y^m and applying p to the resulting equation and using (1.5) we get

$$3LK_m C_{hij} + 3L^2 K p \cdot \dot{\delta}_m C_{hij} + (2LC_{ijm} K_h + h_{ij} K_{hm} + 2LC_{jhm} K_i + h_{jh} K_{im} + 2C_{him} K_j + h_{hi} K_{jm}) = 0.$$

Considering skew symmetric part of the above equation in the indices h and m , we get

$$(2.7) \quad LC_{hij} K_m + h_{jh} K_{im} + h_{hi} K_{jm} - h | m = 0.$$

By virtue of (2.6) and (2.7), we obtain (2.4).

A Riemannian space is characterized by [4]:

$$(2.8) \quad C_{hij} = 0.$$

THEOREM 2.3. *A Finsler space F_n of non-vanishing scalar curvature K with $P_{hij|0} = 0$ is a Riemannian space of constant curvature if*

$$(2.9) \quad 3K K_m^m - K^m K_m = 0,$$

where $K_j^i = g^{im} K_{mj}$, $K^i = g^{im} K_m$.

Proof. Transvecting (2.4) by $h^{ih} = g^{ih} - l^i l^h$ we get

$$(n-1)(3K K_{jm} - K_j K_m) - (3K K_s^s - K^s K_s) h_{jm} = 0$$

which leads to

$$(2.10) \quad 3K K_{jm} - K_j K_m = 0$$

because of (2.9).

Differentiating (2.10) partially with respect to y^h and applying p to the resulting equation, we have

$$(2.11) \quad 3K_h K_{jm} + 3K K_{hjm} - K_{hj} K_m - K_j K_{hm} = 0$$

Equations (2.10) and (2.11) give $K_m K_h K_j + 9K^2 K_{hjm} = 0$ which yields

$$(2.12) \quad K_{hjm} - h | j = 0 \quad K \neq 0,$$

By virtue of (2.2) and (2.12), we get $K_h h_{jm} - h | j = 0$ which shows that

$$(2.13) \quad K_h = 0.$$

On account of remark 2.1 and equations (2.6), (2.8) and (2.13), we have the theorem.

COROLLARY 2.4. *A Finsler space F_n of non-vanishing constant curvature ($K_j = 0, K \neq 0$) with $P_{hij|0} = 0$ is a Riemannian space of constant curvature.*

Proof. Since F_n is of constant curvature, we get $K_j = K_{hj} = 0$. Thus all the conditions of theorem 2.3 are fulfilled. Hence the corollary.

The h -curvature tensor of Rund connection is defined as follows [4]:

$$(2.14) \quad K_{hjk}^i = R_{hjk}^i - C_{hr}^i R_{jk}^r.$$

THEOREM 2.5. *A Finsler space $F_n (n \geq 3)$ of non-vanishing scalar curvature K is a Riemannian space of constant curvature if the h -curvature tensor of Berwald and Rund coincide.*

Proof. From (1.4) and (2.14), we obtain $P_{ij|k}^h - P_{jr}^h P_{ki}^r - j | k = 0$ which implies $P_{ihj|k} - P_{jhr} P_{ki}^r - j | k = 0$. Considering symmetric part of the above equation in i and h , we have

$$(2.15) \quad P_{ihj|k} - \triangleright | k = 0$$

Also from (1.4), we get

$$(2.16) \quad H_{ihjk} + H_{hijk} = -2C_{thr} R_{jk}^r + 2(P_{ihj|k} - j | k)$$

Substitution of (2.15) in (2.16) gives

$$(2.17) \quad H_{ihjk} + H_{hijk} = -2C_{hir} R_{jk}^r$$

By virtue of (2.1a) and (2.1b), we obtain

$$(2.18a) \quad p \cdot H_{jk}^i = LK_j h_k^i / 3 - j | k$$

$$(2.18b) \quad p \cdot H_{hijk} = (Kh_{hj} + K_{hj}/3)h_{ik} - j | k$$

Applying indicatric projection $p \cdot$ on (2.17) and using (2.18a) and (2.18b) we get

$$(2.19) \quad K_{ij} h_{hk} + K_{hj} h_{ik} - j | k = -2LK_j C_{hik} - j | k.$$

Since $P_{ihj|0} = 0$ because of (2.15), using (2.5) and (2.19), we have

$$(2.20) \quad (3K K_{ij} - 2K_i K_j)h_{hk} + (3K K_{hj} - 2K_h K_j)h_{ik} - j | k = 0$$

(2.4) and (2.20) lead to $K_i K_j h_{hk} + K_h K_j h_{ik} - j | k = 0$. Transvecting the last relation by h^{hk} , we get

$$(2.21) \quad (n-1)K_i K_j - K^m K_m h_{ij} = 0$$

Transvecting the relation above by $K^i K^j$, we obtain $(n-2)K^m K_m K^s K_s = 0$, which implies $K^m K_m = 0$. Hence $K_i = 0$ identically. By definition $K_{hj} = 0$ also. Thus all the conditions of theorem 2.3 are satisfied. Hence the Theorem.

T -tensor T_{hijk} is defined by [3]

$$(2.22) \quad T_{hijk} = LC_{hij|k} + l_h C_{ijk} + l_i C_{hjk} + l_j C_{hik} + l_k C_{hij}.$$

Ikeda [2] has proved that Finsler tensor field u^i satisfies

$$(2.23) \quad u^i_{|0}{}^i|_k - u^i_{|k}{}^i|_0 - u^i_{|k}{}^i|_0 = 0$$

THEOREM 2.6. *A Finsler space of non-vanishing scalar curvature with vanishing T -tensor and $P_{hij|0} = 0$ is a Riemannian space of constant curvature if $C_{hij|k|0} = 0$.*

Proof. Since $T = 0$, (2.2) implies

$$(2.24) \quad LC_{hij|k} = -l_h C_{ijk} - l_i C_{hjk} - l_j C_{hik} - l_k C_{hij}.$$

Differentiating (2.24) h -covariantly and transvecting the resulting equation by y and using (2.23), we obtain

$$L(P_{hii|k} - C_{hij|k}) = l_h P_{ijk} - l_i P_{hjk} - l_j P_{hik} - l_k P_{hij}$$

Differentiating the equation above h -covariantly once again and transvecting the resulting equation by y and using (2.23), $P_{hij|0} = 0$ and $C_{hij|h|0} = 0$, we obtain

$$(2.25) \quad P_{hij|k} = 0.$$

From (2.15) – (2.21) and (2.25) we have $K_i = 0$ and consequently $K_{hj} = 0$.

Thus the theorem follows in light of Theorem 2.3.

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