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DIFFERENT KINDS OF THE COVARIANT DIFFERENTATIONS IN RECURRENT FINSLER SPACES

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Abstract. A Finsler spaceis defined to be recurrent if the metric tensor is recurrent. In such a space two ortogonal families of vector fields are defined. Using a family of connection coefficients depending on a parameter, we examine conditions which should be satisfied so that the projections of the metric tensor in the direction of mentioned vector fields are recurrent.

1. Introduction. In this paper the Finsler spaces in which the metric tensor is recurrent i.e. satisfies (2.14)–(2.17) will be examined. In this space m vectorfields $B_a^\alpha(x,\dot{x})$ and $n-m$ vectorfields $N_k^\alpha(x,\dot{x})$ which are linearly indipendent and satisfy (2.1) are given. The vector dx^a and $\dot{x}^a$ are decomposed in the direction of these vectors as it is given by (2.6) and (2.7). We shall suppose that

$$F^a(x^1,\dots,x^n,\dot{x}^1,\dots,\dot{x}^n,u^1,\dots,u^m,v^{m+1},\dots,v^n,\dot{u}^1,\dots,\dot{u}^m,\dot{v}^{m+1},\dots,\dot{v}^n)=0,$$
$$\alpha=1,2,\dots,n$$

any of the solutions of systhem of differential equations (2.6), together with (2.7) define x and $\dot{x}$ as the functions of u, $\dot{u}$, v, $\dot{v}$ in the form

$$x^\alpha=x^\alpha(u^1,\dots,u^m,v^{m+1},\dots,v^n,\dot{u}^1,\dots,\dot{u}^m,\dot{v}^{m+1},\dots,\dot{v}^n)$$
$$\dot{x}^\alpha=\dot{x}^\alpha(u^1,\dots,u^m,v^{m+1},\dots,v^n,\dot{u}^1,\dots,\dot{u}^m,\dot{v}^{m+1},\dots,\dot{v}^n)$$
$$\alpha=1,2,\dots,n$$

In this paper we shall not obtain the above equations but the partial derivatives of tensors with respect to u, v, $\dot{u}$, $\dot{v}$ will be substituted by derivatives with respect to x and $\dot{x}$ (see (4.8)). We shall suppose that the tensor and vector fields in F_n are homogeneous of degree zero in $\dot{x}$. For a vector field $\xi^\alpha(x,\dot{x})$ we have

$$\xi^\alpha(x,\dot{x})=B_a^\alpha(x,\dot{x})\xi^a(x,\dot{x})+N_k^\alpha(x,\dot{x})\xi^k(x,\dot{x}) \tag{1.1}$$

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where $\xi^a = B^a_\alpha \xi^a$, $\xi^k = N^k_\alpha \xi^a$ are also homogeneous functions of degree zero in x. As

$$\xi^x(x,\dot{x}) = \xi^x(u,v,\dot{u},\dot{v}) \quad (x = a \text{ or } x = k)$$

and

$$\lambda \dot{x}^\alpha = B^\alpha_a(x,\dot{x})\lambda\dot{u}^a + N^\alpha_k(x,\dot{x})\lambda\dot{v}^k$$

it follows that

$$\xi^x(u,v,\lambda\dot{u},\lambda\dot{v}) = \xi^x(u,v,\dot{u},\dot{v})$$

and

$$\dot{\delta}_a \xi^x \dot{u}^a + \dot{\delta}_k \xi^x \dot{v}^k = 0 \tag{1.2}$$

where

$$\dot{\delta}_a = \delta/\delta\dot{u}^a, \quad \dot{\delta}_k = \delta/\delta\dot{v}^k.$$

Formulae (1.2) are valid always when instead of ξ^x there are coordinates of any tensor field homogeneous of degree zero in $\dot{x}$.

We shall define different kinds of connection coefficients and covariant differentiations which are generalisations of the induced differential on a subspace of F_n.

For some special cases of (2.6) its solution is a family of subspaces of F_n. Some of these are mentioned here.

If we fix the vector $\dot{x}$ in the equation $dx^\alpha = B^\alpha_a(x,\dot{x})du^a + N^\alpha_k(x,\dot{x})dv^k$ we obtain

$$dx^\alpha = B^\alpha_a(x,\dot{x}_0)du^a + N^\alpha_k(x,\dot{x}_0)dv^k, \quad \alpha = 1,2,\dots,n.$$

For $dv^k = 0$ these equations reduce to $dx^\alpha B^\alpha_a(x)du^a$. These are the differetial equations of the family of subspaces

$$x^\alpha = f^\alpha(u^1,\dots,u^m,C_{m+1},\dots,C_n) \qquad \alpha = 1,2,\dots,n$$

$$\det I = \det\left[\frac{\delta(x^1,x^2,\dots,x^n)}{\delta(u^1,\dots,u^m,C_{m+1},\dots,C_n)}\right]$$

and $C_{m+1},\dots,C_n$ are arbitrary constants. From $\det I \neq 0$ it follows that (1.3) may be written in the form

$$\begin{aligned} u^a &= u^a(x^1,x^2,\dots,x^n), \qquad && a = 1,2,\dots,m \\ C_k &= C_k(x^1,x^2,\dots,x^n), \qquad && k = m+1,\dots,m \end{aligned}$$

Using these equations we obtain from (1.1)

$$\begin{aligned} \delta_a x^\alpha &= \delta_a f^\alpha(u^1,\dots,u^m,C_{m+1},\dots,C_n) = \bar{B}^\alpha_a(u^1,\dots,u^m,C_{m+1},\dots,C_n) \\ &= \bar{B}^\alpha_a[u^1(x^1,\dots,x^n),\dots,C_n(x^1,\dots,x^n)] = B^a_\alpha(x^1,\dots,x^n) \\ &\quad (\delta_a = \delta/\delta u^a). \end{aligned}$$

For all m dimensional subspaces $x^\alpha = f^\alpha(u^1,\dots,u^m,C_{m+1},\dots,C_n)\, B^\alpha_a$ are the tangent vectors and N^α_k are the normal vectors, according to (2.1). For the same fixed $\dot{x}\,(\dot{x} = \dot{x}_0)$ putting $du^a = 0$ we obtain another family of subspaces

$$x^\alpha = g^\alpha(C_1,\dots,C_m v^{m+1},\dots,v^n)$$

for which N_k^α are the tangent vectors and B_a^α are the normal vectors. For every fixed $\dot{x}$ we obtain a similar ortogonal family of subspaces but the induced metric on each subspace is Riemannian. We have

$$g_{ab}(u,v) = g_{\alpha\beta}(x,\dot{x}_0)B_a^\alpha(x,\dot{x}_0)B_b^\beta(x,\dot{x}_0)$$

If we put $v^k = \bar{C}_k$, $k = m+1,\dots,n$ we obtain

$$g_{ab} = g_{ab}(u, \bar{C}_{m+1},\dots,\bar{C}_n),$$

a Riemannian metric on the subspace $x^\alpha = x^\alpha(u^1,\dots,u^m,\bar{C}_{m+1},\dots,\bar{C}_n)$. The more interesting case is when $B_a^\alpha = B_a^\alpha(x)$ on the whole F_n (rank $[B_a^\alpha] = m$). The equations $dx^\alpha = B_a^\alpha(x)du^a$ $(dv^k = 0)$ define a family of subspaces F_m, $x^\alpha = x^\alpha(u^1,\dots,u^m,C_{m+1},\dots,C_n)$ for which $B_a^\alpha(x)$ are the tangent vectors and $N_k^\alpha(x,\dot{x})$ are the normal vectors because $g_{\alpha\beta}(x,\dot{x})B_a^\alpha(x)N_k^\alpha(x,\dot{x}) = 0$. The induced metric is defined as usual by $g_{ab}(u,\dot{u},v,\dot{v}) = g_{\alpha\beta}(x,\dot{x})B_a^\alpha(x)B_b^\beta(x)$ and it is a Finsler metric when $\dot{v} = 0$ (i.e. $\dot{x}^\alpha = B_a^\alpha(x)\dot{u}^a$) and $v^k = \bar{C}_k$ $k = m+1,\dots,n$ on the subspace

$$x^\alpha = x^\alpha(u^1,\dots,u^m,\bar{C}_{m+1},\dots,\bar{C}_n).$$

The situation is similar when $N_k^\alpha = N_k^\alpha(x)$, $k = m+1,\dots,n$ and $B_a^\alpha = B_a^\alpha(x,\dot{x})$, only then N_k^α are the components of tangent vectors of the subspaces and B_a^α are the normal vectors.

The induced differentials $\bar{D}\xi^a$, $\bar{D}\xi^k$ are defined by

$$\bar{D}\xi^a = B_\alpha^a D\xi^\alpha, \qquad \bar{D}\xi^k = N_\alpha^k D\xi^\alpha.$$

For the special case when $\xi^k = 0$ and $\xi^a(u,v,\dot{u},\dot{v}) = \xi^a(u,\dot{u})$, where all v^k are fixed and $\dot{v}^k = 0$ for $k = m+1,\dots,n$ we have the classical case where $\dot{x}^a = B_a^\alpha\dot{u}^a$ and $\xi^\alpha = B_a^\alpha\xi^a$.

In **4** are given conditions when the tensors g_{ab} and g_{nk} will be recurrent with respect to different kinds of covariant differentiation.

2. The induced connection coefficients in a recurrent Finsler space. In the Finsler space F_n the metric function is $F(x,\dot{x})$. Let us define m vector fields $B_a^\alpha(x,\dot{x})$ and n-m vector fields $N_k^\alpha(x,\dot{x})$

$$\alpha,\beta,\gamma,\delta,\varepsilon,\chi,\dots = 1,2,\dots,n$$
$$a,b,c,d,e,f,\dots = 1,2,\dots,m$$
$$k,l,m,n,p,q,\dots = m+1,\dots,n$$

in such a way that these vector fields be linearly independent at each $(x,\dot{x})$ and satisfy the relations

$$g_{\alpha\beta}(x,\dot{x})B_a^\alpha(x,\dot{x})N_k^\alpha(x,\dot{x}) = 0 \tag{2.1}$$

for each $a = 1,2,\dots,m$, $k = m+1,\dots,n$. Let us define

$$\text{a) } g_{ab} = g_{\alpha\beta}B_a^\alpha B_b^\beta \qquad \text{b) } g_{kl} = g_{\alpha\beta}N_k^\alpha N_l^\beta \tag{2.2}$$

$$\text{a) } B_\beta^b = g^{ab}g_{\alpha\beta}B_a^\alpha \qquad \text{b) } N_\alpha^k = g^{km}g_{\alpha\beta}N_m^\beta \tag{2.3}$$

where $g_{\alpha\beta}$, B^α_a and N^β_k are zero degree of homogenity in $\dot{x}$, (g^{ab}) and (g^{km}) are inverse metrices of (g_{ab}) and (g_{km}) respectively. From (2.2) and (2.3) we have

$$\text{(2.4)}\qquad \begin{aligned} &\text{a)} \quad N^k_\alpha N^\alpha_p = g^{kl} g_{\alpha\beta} N^\beta_l N^\alpha_p = g^{kl} g_{lp} = \delta^k_p \\ &\text{b)} \quad B^a_\alpha B^\alpha_b = g^{ac} g_{\alpha\beta} B^\beta_c B^\alpha_b = g^{ac} g_{cb} = \delta^a_b. \end{aligned}$$

As usual

$$\text{(2.5)}\qquad \delta^\alpha_\beta = B^\alpha_a B^a_\beta + N^\alpha_k N^k_\beta.$$

If $\xi^\alpha(x, \dot{x})$ is a vectorfield in F_n homogeneous of degree zero in $\dot{x}$, then

$$\xi^\alpha = B^\alpha_a \xi^a + N^\alpha_k \xi^k.$$

We may write

$$\text{(2.6)}\qquad dx^\alpha = B^\alpha_a du^a + N^\alpha_k dv^k$$

$$\text{(2.7)}\qquad \dot{x}^\alpha = B^\alpha_a \dot{u}^a + N^\alpha_k \dot{v}^k.$$

Let us define the absolute differential which corresponds to the motion from $(x, \dot{x})$ to $(\dot{x} + dx, \dot{x} + d\dot{x})$ by D.

The induced differentials are defined by

$$\text{(2.8)}\qquad \text{a)}\ \bar{D}\xi^a = B^a_\alpha D\xi^\alpha \qquad \text{b)}\ \bar{D}\xi^k = N^k_\alpha D\xi^a$$

and

$$\text{(2.9)}\qquad D\xi^\alpha = B^\alpha_a \bar{D}\xi^a + N^\alpha_k \bar{D}\xi^k.$$

We shall use the notation

$$\text{(2.10)}\qquad l^\alpha = F^{-1}\dot{x}^\alpha = F^{-1}(B^\alpha_a \dot{u}^a + N^\alpha_k \dot{v}^k) = B^\alpha_a l^a + N^a_k l^k$$

where

$$\text{(2.11)}\qquad l^a = F^{-1}\dot{u}^a, \qquad l^k = F^{-1}\dot{v}^k.$$

From (2.9) we have

$$\text{(2.12)}\qquad Dl^\alpha = B^\alpha_a \bar{D}l^a + N^\alpha_k \bar{D}l^k$$

where

$$Dl^\alpha = dl^\alpha + \Gamma^*_{0\,\gamma}{}^{\alpha} dx^\gamma + A_{0\,\gamma}{}^{\alpha} Dl^\gamma.$$

We shall suppose that the metric tensor is determined by

$$\text{(2.13)}\qquad g_{\alpha\beta}(x, \dot{x}) = 2^{-1}\dot{\delta}_\alpha \dot{\delta}_\beta F^2(x, \dot{x})$$

and that the space F_n is recurrent, i.e.

$$\text{(2.14)}\qquad g_{\alpha\beta|\gamma} = \lambda_\gamma(x, \dot{x}) g_{\alpha\beta}.$$

$$\text{(2.15)}\qquad g_{\alpha\beta}\big|_\gamma = \mu_\gamma(x, \dot{x}) g_{\alpha\beta}.$$

As

$$Dg_{\alpha\beta} = g_{\alpha\beta|\gamma} dx^\gamma + g_{\alpha\beta}\big|_\gamma Dl^\gamma. \tag{2.16}$$

from (2.14) and (2.15) we obtain

$$Dg_{\alpha\beta} = K(x, \dot{x}, dx, Dl) g_{\alpha\beta}. \tag{2.17}$$

where

$$K(x, \dot{x}, dx, Dl) = \lambda_\gamma(x, \dot{x}) dx^\gamma + \mu_\gamma(x, \dot{x}) Dl^\gamma. \tag{2.18}$$

The absolute differential of $g_{\alpha\beta}$ is

$$Dg_{\alpha\beta} = dg_{\alpha\beta} - (\Gamma^{*\delta}_{\alpha\gamma} g_{\delta\beta} + \Gamma^{*\delta}_{\beta\gamma} g_{\alpha\delta}) dx^\gamma - (A_\alpha{}^\delta{}_\gamma g_{\delta\beta} + A_\beta{}^\delta{}_\gamma g_{\alpha\delta}) Dl^\gamma. \tag{2.19}$$

The connection coefficients are determined in [2] under conditions $\Gamma^*_{\alpha\beta\gamma} = \Gamma^*_{\gamma\beta\alpha}$ and $A_{\alpha\beta\gamma} = A_{\beta\alpha\gamma}$, but in [3] under conditions $\Gamma^*_{\alpha\beta\gamma} = \Gamma^*_{\gamma\beta\alpha}$, $A_{\alpha\beta\gamma} = A_{\gamma\beta\alpha}$.

In this paper we shall use two last conditions.

The vector dx^γ and Dl^γ $(\gamma = 1, 2, \dots, n)$ are not linearly indipendent. As is known that in the non-recurrent Finsler space from $g_{\alpha\beta} l^\alpha l^\beta = 1$ it follows that $l_\alpha Dl^\alpha = 0$. In a recurrent Finsler space from $g_{\alpha\beta} l^\alpha l^\beta = 1$ we obtain using (2.14)–(2.16)

$$\lambda_\beta dx^\beta + (\mu_\beta + 2l_\beta) Dl^\beta = 0. \tag{2.20}$$

In [3] the induced connection coefficients are determined, but (2.20) was not taken into account. Here this will be done and we obtain a family of connection coefficients depending on some parametres. As in [3] we shall write

$$DB^\alpha_a = \bar{w}^d_a(d) B^\alpha_d + \bar{w}^m_a(d) N^\alpha_m \tag{2.21}$$

$$DN^\alpha_k = \bar{w}^d_k(d) B^\alpha_d + \bar{w}^m_k(d) N^\alpha_m \tag{2.22}$$

where

$$\begin{gathered} \bar{w}^x_y(d) = \bar{\Gamma}^{*x}_{yb} du^b + \bar{\Gamma}^{*x}_{yk} dv^k + \bar{A}_y{}^x{}_b \bar{D}l^b + \bar{A}_y{}^x{}_k \bar{D}l^k \\ x = d \text{ or } x = m, \qquad y = a \text{ or } y = k. \end{gathered} \tag{2.23}$$

From (2.1) and (2.17) we have

$$D(g_{\alpha\beta} B^\alpha_a N^\beta_k) = g_{\alpha\beta}(DB^\alpha_a) N^\beta_k + g_{\alpha\beta} B^\alpha_a DN^\beta_k = 0.$$

Substituting (2.21) and (2.22) into the above equation and using (2.2) we obtain

$$\bar{w}_{ak} = -\bar{w}_{ka} \Leftrightarrow g_{km} \bar{w}^m_a = -g_{ab} \bar{w}^b_k, \tag{2.24}$$

the same equation as in a non-recurrent Finsler space. If we express DB^α_a by the connection coefficients of the space F_n and use (2.6) and (2.12) we get

$$DB^\alpha_a = (B^\alpha_{a|\beta} du^b + B^\alpha_a\big|_\beta \bar{D}l^b) B^\beta_b + (B^\alpha_{a|\beta} dv^k + B^\alpha_a\big|_\beta \bar{D}l^k) N^\beta_k, \tag{2.25}$$

where

$$
\text{(2.26)} \quad
\begin{aligned}
&\text{a)}\ B^{\alpha}_{a|\beta} = \delta_\beta B^{\alpha}_{a} - \dot{\delta}_\gamma B^{\alpha}_{a}\Gamma^{*\gamma}_{\ \beta} + \Gamma^{*\ \alpha}_{\gamma\beta}B^{\gamma}_{a} \qquad (\Gamma^{*\gamma}_{\ \beta} = \Gamma^{*\ \gamma}_{\alpha\beta}\dot{x}^\alpha) \\
&\text{b)}\ B^{\alpha}_{a}\big|_{\beta} = F\dot{\delta}_\delta B^{\alpha}_{a}(\delta^{\delta}_{\beta} - A_{0\ \beta}^{\ \delta}) + A_{\delta\ \beta}^{\ \alpha}B^{\delta}_{a} \quad (A_{0\ \beta}^{\ \delta} = A_{\alpha\ \beta}^{\ \delta}l^\alpha)
\end{aligned}
$$

If we substitute (2.6) and (2.12) into (2.20) using the notations

$$
\text{(2.27)} \qquad \lambda_b = B^{\beta}_{b}\lambda_\beta, \quad \lambda_k = N^{\beta}_{k}\lambda_\beta, \quad \mu_c = B^{\beta}_{b}\mu_\beta, \quad \mu_k = N^{\beta}_{k}\mu_\beta,
$$

we obtain

$$
\text{(2.28)} \qquad 0 = \theta^{\alpha}_{a}(x,\dot{x})[\lambda_b du^b + \lambda_k dv^k + (\mu_b + 2l_b)\bar{D}l^b + (\mu_k + 2l_k)\bar{D}l^k]
$$

where θ^{α}_{a} is any parameter homogeneous of degree zero in $\dot{x}$. If we equate the right hand side of (2.21) with the sum of the right-hand sides of (2.25) and (2.28) we get an equation where on the both sides terms with factors du^b, dv^k, $\bar{D}l^b$ and $\bar{D}l^k$ are present. Equating the corresponding coefficients we obtain

$$
\begin{aligned}
du^b &: \quad \bar{\Gamma}^{*\ d}_{a\ b}B^{\alpha}_{d} + \bar{\Gamma}^{*\ m}_{a\ b}N^{\alpha}_{m} = B^{\alpha}_{a|\beta}B^{\beta}_{b} + \theta^{\alpha}_{a}\lambda_b \\
dv^k &: \quad \bar{\Gamma}^{*\ d}_{a\ k}B^{\alpha}_{d} + \bar{\Gamma}^{*\ m}_{a\ k}N^{\alpha}_{m} = B^{\alpha}_{a|\beta}N^{\beta}_{k} + \theta^{\alpha}_{a}\lambda_k \\
\bar{D}l^b &: \quad \bar{A}_{a\ b}^{\ d}B^{\alpha}_{d} + \bar{A}_{a\ b}^{\ m}N^{\alpha}_{m} = B^{\alpha}_{a}\big|_{\beta}B^{\beta}_{b} + \theta^{\alpha}_{a}(\mu_b + 2l_b) \\
\bar{D}l^k &: \quad \bar{A}_{a\ k}^{\ d}B^{\alpha}_{d} + \bar{A}_{a\ k}^{\ m}N^{\alpha}_{m} = B^{\alpha}_{a}\big|_{\beta}N^{\beta}_{k} + \theta^{\alpha}_{a}(\mu_k + 2l_k)
\end{aligned}
$$

Multiplying the above equations first by $g_{\alpha\gamma}B^{\gamma}_{c}$ then by $g_{\alpha\gamma}N^{\gamma}_{n}$ and using the notation $\theta_{ac} = \theta^{\alpha}_{a}g_{\alpha\gamma}B^{\gamma}_{c}$, $\theta_{an} = \theta^{\alpha}_{a}g_{\alpha\gamma}N^{\gamma}_{n}$ we obtain

$$
\text{(2.29)} \quad
\begin{aligned}
&\text{(a)} \quad \bar{\Gamma}^{*}_{acb} = g_{\alpha\gamma}B^{\gamma}_{c}B^{\beta}_{b}B^{\alpha}_{a|\beta} + \theta_{ac}\lambda_b \\
&\text{(b)} \quad \bar{\Gamma}^{*}_{ack} = g_{\alpha\gamma}B^{\gamma}_{c}N^{\beta}_{k}B^{\alpha}_{a|\beta} + \theta_{ac}\lambda_k \\
&\text{(c)} \quad \bar{A}_{acb} = g_{\alpha\gamma}B^{\gamma}_{c}B^{\beta}_{b}B^{\alpha}_{a}\big|_{\beta} + \theta_{ac}(\mu_b + 2l_b) \\
&\text{(d)} \quad \bar{A}_{ack} = g_{\alpha\gamma}B^{\gamma}_{c}N^{\beta}_{k}B^{\alpha}_{a}\big|_{\beta} + \theta_{ac}(\mu_k + 2l_k) \\
&\text{(e)} \quad \bar{\Gamma}^{*}_{anb} = g_{\alpha\gamma}N^{\gamma}_{n}B^{\beta}_{b}B^{\alpha}_{a|\beta} + \theta_{an}\lambda_b \\
&\text{(f)} \quad \bar{\Gamma}^{*}_{ank} = g_{\alpha\gamma}N^{\gamma}_{n}N^{\beta}_{k}B^{\alpha}_{a|\beta} + \theta_{an}\lambda_k \\
&\text{(g)} \quad \bar{A}_{anb} = g_{\alpha\gamma}N^{\gamma}_{n}B^{\beta}_{b}B^{\alpha}_{a}\big|_{\beta} + \theta_{an}(\mu_b + 2l_b) \\
&\text{(h)} \quad \bar{A}_{ank} = g_{\alpha\gamma}N^{\gamma}_{n}N^{\beta}_{k}B^{\alpha}_{a}\big|_{\beta} + \theta_{an}(\mu_k + 2l_k).
\end{aligned}
$$

In a similar manner using the expression for DN^{α}_{k} and the notations

$$
\nu_{kc} = \nu^{\alpha}_{k}g_{\alpha\gamma}B^{\gamma}_{c}, \qquad \nu_{kn} = \nu^{\alpha}_{k}g_{\alpha\gamma}N^{\gamma}_{n},
$$

where $\nu_k^\alpha(x,\dot{x})$ is any parameter homogeneous of degree zero in $\dot{x}$ we obtain

$$(2.30)\quad \begin{aligned}
&\text{(a)} && \bar{\Gamma}^*_{kcb} = g_{\alpha\gamma}B_c^\gamma B_b^\beta N^\alpha_{k|\beta} + \nu_{kc}\lambda_b\\
&\text{(b)} && \bar{\Gamma}^*_{kcl} = g_{\alpha\gamma}B_c^\gamma N_l^\beta N^\alpha_{k|\beta} + \nu_{kc}\lambda_l\\
&\text{(c)} && \bar{A}_{kcb} = g_{\alpha\gamma}B_c^\gamma B_b^\beta N^\alpha_k\big|_\beta + \nu_{kc}(\mu_b + 2l_b)\\
&\text{(d)} && \bar{A}_{kcl} = g_{\alpha\gamma}B_c^\gamma N_l^\beta N^\alpha_k\big|_\beta + \nu_{kc}(\mu_l + 2l_l)\\
&\text{(e)} && \bar{\Gamma}^*_{knb} = g_{\alpha\gamma}N_n^\gamma B_b^\beta N^\alpha_{k|\beta} + \nu_{kn}\lambda_b\\
&\text{(f)} && \bar{\Gamma}^*_{knl} = g_{\alpha\gamma}N_n^\gamma N_l^\beta N^\alpha_{k|\beta} + \nu_{kn}\lambda_l\\
&\text{(g)} && \bar{A}_{knb} = g_{\alpha\gamma}N_n^\gamma B_b^\beta N^\alpha_k\big|_\beta + \nu_{kn}(\mu_b + 2l_b)\\
&\text{(h)} && \bar{A}_{knl} = g_{\alpha\gamma}N_n^\gamma N_l^\beta N^\alpha_k\big|_\beta + \nu_{kn}(\mu_l + 2l_l)
\end{aligned}$$

The connection coefficients obtained in [3] are the special case of those obtained here, when we take $\theta_a^\alpha = 0$ and $\nu_k^\alpha = 0$.

The parametres θ_a^α and ν_k^α cannot be chosen arbitrarily because of (2.24), from which we obtain

$$\begin{aligned}
&g_{nk}(\bar{\Gamma}^{*}{}_a{}^n{}_b du^b + \bar{\Gamma}^{*}{}_a{}^n{}_l dv^l + \bar{A}_a{}^n{}_b \bar{D}l^b + \bar{A}_a{}^n{}_l \bar{D}l^l)\\
&\quad = -g_{ad}(\bar{\Gamma}^{*}{}_k{}^d{}_b du^b + \bar{\Gamma}^{*}{}_k{}^d{}_l dv^l + \bar{A}_k{}^d{}_b \bar{D}l^b + \bar{A}_k{}^d{}_l \bar{D}l^l).
\end{aligned}$$

From the above relation it follows

$$(2.31)\quad \begin{aligned}
\bar{\Gamma}^*_{akb} &= -\bar{\Gamma}^*_{kab} & \qquad \bar{\Gamma}^*_{akl} &= -\bar{\Gamma}^*_{kal}\\
\bar{A}_{akb} &= -\bar{A}_{kab} & \qquad \bar{A}_{akl} &= -\bar{A}_{kal}.
\end{aligned}$$

Substituting the connection coefficients from (2.29) and (2.30) into (2.31) and using the relation

$$g_{\alpha\gamma}B^\alpha_{a|\beta}N_k^\gamma + g_{\alpha\gamma}B_a^\alpha N^\gamma_{k|\beta} = 0$$

and the similar one with $\big|_\beta$ we obtain

$$(2.32)\qquad \theta_{ak} = -\nu_{ka}.$$

3. Different kinds of covariant differentiation. From (2.7) and (2.12) we have

$$(3.1)\qquad Dl^\alpha = B_a^\alpha \bar{D}l^a + N_k^\alpha \bar{D}l^k = (DB_a^\alpha)l^a + (DN_m^\alpha)l^m + B_a^\alpha dl^a + N_k^\alpha dl^k.$$

In we substitute from (2.21) and (2.22) the expression for DB_a^α and DN_m^α using the notations of [4]

$$(3.2)\qquad \bar{\Gamma}^{*}{}_0{}^x{}_y = \bar{\Gamma}^{*}{}_a{}^x{}_y l^a + \bar{\Gamma}^{*}{}_m{}^x{}_y l^m \qquad x = d \text{ or } x = m$$

$$(3.3)\qquad A_0{}^x{}_y = \bar{A}_a{}^x{}_y l^a + \bar{A}_m{}^x{}_y l^m \qquad y = b \text{ or } y = k$$

we obtain

$$\bar{D}l^d = dl^d + \bar{\Gamma}^{*}{}_{0}{}^{d}{}_{b} du^b + \bar{\gamma}^{*}{}_{0}{}^{d}{}_{k} dv^k + \bar{A}_{0}{}^{d}{}_{b} \bar{D}l^b + \bar{A}_{0}{}^{d}{}_{k} \bar{D}l^k \tag{3.4}$$

$$\bar{D}l^m = dl^m + \bar{\Gamma}^{*}{}_{0}{}^{m}{}_{b} du^b + \bar{\gamma}^{*}{}_{0}{}^{m}{}_{k} dv^k + \bar{A}_{0}{}^{m}{}_{b} \bar{D}l^b + \bar{A}_{0}{}^{m}{}_{k} \bar{D}l^k. \tag{3.5}$$

From (2.11) we have

$$dl^d = F^{-1} d\dot{u}^d + \dot{u}^d dF^{-1}, \qquad dl^m = F^{-1} d\dot{v}^m + \dot{v}^m dF^{-1}.$$

If we substitute the above equations into (3.4) and (3.5) we get

$$d\dot{u}^d = -F\bar{\Gamma}^{*}{}_{0}{}^{d}{}_{b} du^b - F\bar{\Gamma}^{*}{}_{0}{}^{d}{}_{k} dv^k + F(\delta^d_b - A_{0}{}^{d}{}_{b})\bar{D}l^b - F\bar{A}_{0}{}^{d}{}_{k}\bar{D}l^k + \dot{u}^d F^{-1} dF \tag{3.6}$$

$$d\dot{v}^m = -F\bar{\Gamma}^{*}{}_{0}{}^{m}{}_{b} du^b - F\bar{\Gamma}^{*}{}_{0}{}^{m}{}_{k} dv^k - FA_{0}{}^{m}{}_{b}\bar{D}l^b + F(\delta^m_k - \bar{A}_{0}{}^{m}{}_{k})\bar{D}l^k + \dot{v}^m F^{-1} dF. \tag{3.7}$$

For any vectorfield $\xi^\alpha(x, \dot{x})$ in F_n we have from (2.9)

$$D\xi^\alpha = B^\alpha_a \bar{D}\xi^a + N^\alpha_k \bar{D}\xi^b = (DB^\alpha_a)\xi^a + (DN^\alpha_k)\xi^k + B^\alpha_a d\xi^a + N^\alpha_k d\xi^k. \tag{3.8}$$

Substituting $d\dot{u}^d$ and $d\dot{v}^m$ from (3.6) and (3.7) into

$$d\xi^a = \delta_d \xi^a du^d + \dot{\delta}_d \xi^k d\dot{u}^d + \delta_m \xi^a dv^m + \dot{\delta}_m \xi^a d\dot{v}^m \tag{3.9}$$

$$d\xi^k = \delta_d \xi^k du^d + \dot{\delta}_d \xi^k d\dot{u}^d + \delta_m \xi^k dv^m + \dot{\delta}_m \xi^k d\dot{v}^m \tag{3.10}$$

and so obtained $d\xi^a$ and $d\xi^k$ into (3.8) we obtain

$$\begin{aligned} D\xi^\alpha =& B^\alpha_a(\xi^a_{\top c} du^c + \xi^a_{\top m} dv^m + \xi^{\overline{a}|}{}_{c}\ \bar{D}l^c + \xi^{\overline{a}|}{}_{m} \bar{D}l^m) \\ &+ N^\alpha_n(\xi^n_{\top c} du^c + \xi^n_{\top m} dv^m + \xi^{\overline{n}|}{}_{c}\ \bar{D}l^c + \xi^{\overline{n}|}{}_{m} \bar{D}l^m) \\ &+ B^\alpha_a F^{-1} dF(\dot{\delta}_d \xi^a \dot{u}^d + \dot{\delta}_m \xi^a \dot{v}^m) + N^\alpha_n F^{-1} dF(\dot{\delta}_d \xi^n \dot{u}^d + \dot{\delta}_m \xi^n \dot{v}^m) \end{aligned} \tag{3.11}$$

where

$$\begin{aligned} &\text{(a)} \quad \xi^x_{\top c} &&= \delta_c \xi^x - F\dot{\delta}_d \xi^x \bar{\Gamma}^{*}{}^{d}_{0c} - F\dot{\delta}_k \xi^x \bar{\Gamma}^{*}{}^{k}_{0c} + \bar{\Gamma}^{*}{}^{x}_{bc}\xi^b + \bar{\Gamma}^{*}{}^{x}_{kc}\xi^k \\ &\text{(b)} \quad \xi^x_{\top m} &&= \delta_m \xi^x - F\dot{\delta}_d \xi^x \bar{\Gamma}^{*}{}^{d}_{0m} - F\dot{\delta}_k \xi^x \bar{\Gamma}^{*}{}^{k}_{0m} + \bar{\Gamma}^{*}{}^{x}_{bm}\xi^b + \bar{\Gamma}^{x}_{km}\xi^k \\ &\text{(c)} \quad \xi^{\overline{x}|}{}_{c} &&= F\dot{\delta}_d \xi^x(\delta^d_c - \bar{A}_{0}{}^{d}{}_{c}) - F\dot{\delta}_k \xi^x \bar{A}_{0}{}^{k}{}_{c} + \bar{A}_{b}{}^{x}{}_{c}\xi^b + \bar{A}_{k}{}^{x}{}_{c}\xi^k \\ &\text{(d)} \quad \xi^{\overline{x}|}{}_{m} &&= -F\dot{\delta}_d \xi^x \bar{A}_{0}{}^{d}{}_{m} + F\dot{\delta}_k \xi^x(\delta^k_m - \bar{A}_{0}{}^{k}{}_{m}) + \bar{A}_{b}{}^{x}{}_{m}\xi^b + \bar{A}_{k}{}^{x}{}_{m}\xi^k \end{aligned} \tag{3.12}$$

$x = a$ or $x = n$.

Using the homogenity conditions for ξ^a and ξ^k we have (see (1.2))

$$\dot{\delta}_d \xi^x \dot{u}^d + \dot{\delta}_m \xi^x \dot{v}^m = 0 \qquad (x = a \text{ or } x = k),$$

so the last two terms in (3.11) are equal to zero and we obtain

$$D\xi^\alpha = B^\alpha_a \bar{D}\xi^a + N^\alpha_n \bar{D}\xi^n, \tag{3.13}$$

where

$$\bar{D}\xi^a = \xi^a_{\top c} du^c + \xi^a_{\top m} dv^m + \xi^{\overline{a}|}{}_{c}\ \bar{D}l^c + \xi^{\overline{a}|}{}_{m} \bar{D}l^m \tag{3.14}$$

$$\bar{D}\xi^n = \xi^n_{\top c} du^c + \xi^n_{\top m} dv^m + \xi^{\overline{n}|}{}_{c}\ \bar{D}l^c + \xi^{\overline{a}|}{}_{m} \bar{D}l^m. \tag{3.15}$$

For the metric tensor $g_{\alpha\beta}$ the above formulae have the form:

$$
\begin{aligned}
Dg_{\alpha\beta} = & B^{ab}_{\alpha\beta}(g_{ab\top c}du^c + g_{ab\top k}dv^k + g_{ab\overline{|}c}\ \bar{D}l^c + g_{ab\overline{|}k}\ \bar{D}l^k \\
& + D^{nt}_{\alpha\beta}g(g_{nt\top c}du^c + g_{nt\top k}dv^k + g_{nt\overline{|}c}\ \bar{D}l^c + g_{nt\overline{|}k}\ \bar{D}l^k) \\
& + \hat{\theta}_{\alpha\beta}(\lambda_c du^c + \lambda_k dv^k + (\mu_c + 2l_c)\bar{D}l^c + (\mu_k + l_k)\bar{D}l^k),
\end{aligned}
\tag{3.16}
$$

where $\hat{\theta}_{\alpha\beta} = \hat{\theta}_{\alpha\beta}(x, \dot{x})$ is a tensor homogenesous of degree zero in $\dot{x}$ and

$$
\begin{aligned}
g_{ab\top x} &= \delta_x g_{ab} - F\dot{\delta}_d g_{ab}\bar{\Gamma}^{*d}_{0x} - F\dot{\delta}_m g_{ab}\bar{\Gamma}^{*m}_{0x} - g_{db}\bar{\Gamma}^{ast\ d}_{ax} - g_{bd}\bar{\Gamma}^{*d}_{ax} \\
&\qquad\qquad\qquad\qquad (x = c \text{ or } x = k), \\
g_{nt\top x} &= \delta_x g_{nt} - F\dot{\delta}_d g_{nt}\bar{\Gamma}^{*d}_{0x} - F\dot{\delta}_m g_{nt}\bar{\Gamma}^{*m}_{0x} - g_{mt}\bar{\Gamma}^{ast\ m}_{nx} - g_{nm}\bar{\Gamma}^{*m}_{tx} \\
&\qquad\qquad\qquad\qquad (x = c \text{ or } x = k), \\
g_{ab\overline{|}c} &= F\dot{\delta}_d g_{ab}(\delta^d_c - \bar{A}^d_{0c}) - F\dot{\delta}_m g_{ab}\bar{A}^m_{0c} - g_{db}\bar{A}^d_{ac} - g_{ad}\bar{A}^d_{bc}, \\
g_{ab\overline{|}k} &= -F\dot{\delta}_d g_{ab}\bar{A}^d_{0k} + F\dot{\delta}_m g_{ab}(\delta^m_k - \bar{A}^m_{0k}) - g_{db}\bar{A}^d_{ak} - g_{ad}\bar{A}^d_{bk}, \\
g_{nt\overline{|}c} &= F\dot{\delta}_d g_{nt}(\delta^d_c - \bar{A}^d_{0c}) - F\dot{\delta}_m g_{nt}\bar{A}^m_{0c} - g_{mt}\bar{A}^m_{nc} - g_{nm}\bar{A}^m_{tc}, \\
g_{nt\overline{|}k} &= -F\dot{\delta}_d g_{nt}\bar{A}^d_{0k} + F\dot{\delta}_m g_{nt}(\delta^m_k - \bar{A}^m_{0k}) - g_{mt}\bar{A}^m_{nk} - g_{nm}\bar{A}^m_{tk}.
\end{aligned}
\tag{3.17}
$$

The above relations are valid only on condition that

$$
\dot{\delta}_c g_{ab}\dot{u}^c + \dot{\delta}_k g_{ab}\dot{v}^k = 0 \tag{3.18}
$$

which is satisfied because $g_{ab}(u, v, \dot{u}, \dot{v})$, $g_{nt}(u, v, \dot{u}, \dot{v})$ are homogeneous of degree zero in $\dot{u}$ and $\dot{v}$.

In [3] $\bar{D}l^a$ and $\bar{D}l^k$ are defined by

$$
\text{a) } \bar{D}l^a = dl^a + \bar{\Gamma}^{*a}_{0c}du^c + \bar{\Gamma}^{*a}_{0k}dv^k, \qquad \text{b) } \bar{D}l^k = dl^k + \bar{\Gamma}^{*k}_{0c}du^c + \bar{\Gamma}^{*k}_{0l}dv^l. \tag{3.19}
$$

These formulae are different from (3.4) and (3.5) of present paper, but they may be obtained as a special case of (3.4) and (3.5) if we put

$$
\theta^\alpha_a = 0, \quad \nu^\alpha_k = 0, \quad \bar{A}^d_{0b}\bar{D}l^b + \bar{A}^d_{0k}\bar{D}l^k = 0, \quad \bar{A}^m_{0b}\bar{D}l^b + \bar{A}^m_{0k}\bar{D}l^k = 0.
$$

Only under these conditions (3.19) and (2.12) are consistent. The conditions $\bar{A}^x_{0b}\bar{D}l^b + \bar{A}^x_{0k}\bar{D}l^k = 0$ $x = d$, or $x = m$ are equivalent (according to (3.12) and (3.16) of [3]) to $B^d_\alpha Dl^\beta(B^\alpha_a|_\beta l^a + N^\alpha_k|_\beta l^k) = 0$ and $N^m_\alpha Dl^\beta(B^\alpha_a|_\beta l^a + N^\alpha_n|_\beta l^n) = 0$.

Both conditions are satiafied when

$$
Dl^\beta(B^\alpha_a|_\beta l^a + N^\alpha_k|_\beta l^k) = 0. \tag{3.20}
$$

If we define $\bar{D}l^a$ and $\bar{D}l^m$ as we have done here by (3.4) and (3.5) then we do not have the restricted condition (3.20). $g_{ab\overline{|}c}$, $g_{ab\overline{|}k}$, $g_{nt\overline{|}c}$, $g_{nt\overline{|}k}$ given by (3.17) have more terms containing A^x_{0y} then the corresponding formulae (4.15) of [3].

If F_n satisfies $g_{\alpha\beta}\big|_\gamma = 0$ i.e. $\mu_\gamma = 0$ then $A^\alpha_{0\beta} = 0$ and the condition (3.20) reduces to $Dl^\beta[(\dot{\delta}_\beta B^\alpha_a)l^a + \dot{\delta}_\beta N^\alpha_k)l^k] = 0$. Using (2.6) and (2.7) we obtain

$$Dg_{\alpha\beta} = B^\delta_c g_{\alpha\beta|\delta} du^c + N^\delta_k g_{\alpha\beta|\delta} dv^k + B^\delta_c g_{\alpha\beta|\delta} \bar{D} l^c + N^\delta_k g_{\alpha\beta|\delta} \bar{D} l^k. \tag{3.21}$$

Comparing (3.21) with (3.16) we get

$$B^\delta_c g_{\alpha\beta|\delta} = B^{ab}_{\alpha\beta} g_{ab\top c} + N^{nt}_{\alpha\beta} g_{nt\top c} + \hat{\theta}_{\alpha\beta}\lambda_c \tag{3.22}$$

$$N^\delta_k g_{\alpha\beta|\delta} = B^{ab}_{\alpha\beta} g_{ab\top k} + N^{nt}_{\alpha\beta} g_{nt\top k} + \hat{\theta}_{\alpha\beta}\lambda_k \tag{3.23}$$

$$B^\delta_c g_{\alpha\beta}\big|_\delta = B^{ab}_{\alpha\beta} g_{ab}\overline{\big|}_c + N^{nt}_{\alpha\beta} g_{nt}\overline{\big|}_c + \hat{\theta}_{\alpha\beta}(\mu_c + 2l_c) \tag{3.24}$$

$$N^\delta_k g_{\alpha\beta}\big|_\delta = B^{ab}_{\alpha\beta} g_{ab}\overline{\big|}_k + N^{nt}_{\alpha\beta} g_{nt}\overline{\big|}_k + \hat{\theta}_{\alpha\beta}(\mu_k + 2l_k). \tag{3.25}$$

Multiplying (3.22) by B^c_γ and (3.23) by N^k_γ and adding these relations we get

$$g_{\alpha\beta|\gamma} = B^{abc}_{\alpha\beta\gamma} g_{ab\top c} + B^{ab}_{\alpha\beta} N^k_\gamma g_{ab\top k} + N^{nt}_{\alpha\beta} B^c_\gamma g_{nt\top c} + N^{ntk}_{\alpha\beta\gamma} g_{nt\top k} + \lambda_\gamma \hat{\theta}_{\alpha\beta}. \tag{3.26}$$

By the same process from (3.24) and (3.25) we obtain

$$\begin{aligned} g_{\alpha\beta}\big|_\gamma = B^{abc}_{\alpha\beta\gamma} g_{ab}\overline{\big|}_c &+ B^{ab}_{\alpha\beta} N^k_\gamma g_{ab}\overline{\big|}_k + N^{nt}_{\alpha\beta} B^c_\gamma g_{nt}\overline{\big|}_c \\ &+ N^{ntk}_{\alpha\beta\gamma} g_{nt\top k} + \hat{\theta}_{\alpha\beta}(\mu_\gamma + 2l_\gamma). \end{aligned} \tag{3.27}$$

THEOREM 3.1. *The necessary and sufficient conditions for* $g_{\alpha\beta|\gamma} = \lambda_\gamma g_{\alpha\beta}$ *are*

$$\begin{aligned} &\text{(a)}\ \ g_{ab|c} = \lambda_c(g_{ab} - \hat{\theta}_{ab}) \qquad &&\text{(c)}\ \ g_{nt|c} = \lambda_c(g_{nt} - \hat{\theta}_{nt}) \\ &\text{(b)}\ \ g_{ab|k} = \lambda_k(g_{ab} - \hat{\theta}_{ab}) \qquad &&\text{(d)}\ \ g_{nt|k} = \lambda_k(g_{nt} - \hat{\theta}_{nt}) \end{aligned} \tag{3.28}$$

where

$$\lambda_\gamma = B^c_\gamma \lambda_c + N^k_\gamma \lambda_k, \quad \hat{\theta}_{\alpha\beta} = \hat{\theta}_{ab} B^{ab}_{\alpha\beta} + \hat{\theta}_{nk} N^{nk}_{\alpha\beta}, \quad \hat{\theta}_{alpha\beta} B^\alpha_a N^\beta_k = 0. \tag{3.29}$$

Proof. Substituting (3.28) and (3.29) into (3.26) we get

$$\begin{aligned} g_{\alpha\beta|\gamma} =& B^{ab}_{\alpha\beta}(B^c_\gamma\lambda_c + N^k_\gamma\lambda_k)(g_{ab} - \hat{\theta}_{ab}) \\ &+ N^{nt}_{\alpha\beta}(B^c_\gamma\lambda_c + N^k_\gamma\lambda_k)(g_{nt} - \hat{\theta}_{nt}) + \lambda_\gamma\hat{\theta}_{\alpha\beta} = \lambda_\gamma g_{\alpha\beta}. \end{aligned}$$

On the other hand if $g_{\alpha\beta|\gamma} = \lambda_\gamma g_{\alpha\beta}$ then from $N^k_\alpha B^\alpha_c = 0$, $B^\alpha_a B^c_\alpha = \delta^c_a$, $N^\alpha_k N^n_\alpha = \delta^n_k$ and (3.26) we obtain

$$\begin{aligned} g_{\alpha\beta|\gamma} B^{\alpha\beta\gamma}_{fed} &= \delta^a_f \delta^b_e \delta^c_d g_{ab\top c} + \lambda_d \hat{\theta}_{fe} \Rightarrow g_{fe\top d} = \lambda_d(g_{fe} - \hat{\theta}_{fe}) \\ g_{\alpha\beta|\gamma} B^\alpha_f B^\beta_e N^\gamma_n &= \delta^a_f \delta^b_e \delta^k_n g_{ab\top k} + \lambda_n \hat{\theta}_{fe} \Rightarrow g_{fe\top n} = \lambda_n(g_{fe} - \hat{\theta}_{fe}) \\ g_{\alpha\beta|\gamma} N^\alpha_k N^\beta_l B^\gamma_d &= \delta^n_k \delta^t_l \delta^c_d g_{nt\top c} + \lambda_d \hat{\theta}_{kl} \Rightarrow g_{kl\top d} = \lambda_d(g_{kl} - \hat{\theta}_{kl}) \\ g_{\alpha\beta|\gamma} N^\alpha_p N^\beta_l N^\gamma_m &= \delta^n_p \delta^t_l \delta^k_m g_{nt\top k} + \lambda_m \hat{\theta}_{pl} \Rightarrow g_{pl\top m} = \lambda_m(g_{pl} - \hat{\theta}_{pl}). \end{aligned}$$

THEOREM 3.2. *The necessary and sufficient conditions for* $g_{\alpha\beta}\big|_{\gamma} = \mu_{\gamma} g_{\alpha\beta}$ *are*

$$\text{(3.30)}\quad \begin{aligned} &\text{(a)}\ g_{ab}\overline{\big|}_{c} = \mu_c g_{ab} - (\mu_c + 2l_c)\hat{\theta}_{ab}, \qquad g_{nt}\overline{\big|}_{c} = \mu_c g_{nt} - (\mu_c + 2l_c)\hat{\theta}_{nt} \\ &\text{(b)}\ g_{ab}\overline{\big|}_{k} = \mu_k g_{ab} - (\mu_k + 2l_k)\hat{\theta}_{ab}, \qquad g_{nt}\overline{\big|}_{k} = \mu_k g_{nt} - (\mu_k + 2l_k)\hat{\theta}_{nt} \end{aligned}$$

where

$$\text{(3.31)}\qquad \mu_{\gamma} = B^c_{\gamma}\mu_c + N^k_{\gamma}\mu_k$$

The proof follows from (3.27) using the similar method as in the previous Theorem.

4. Connection between the partial differentiation with respect to different variables. In formulae (3.12) we can not calculate $\delta_c\xi^x$, $\delta_m\xi^x$, $\dot{\delta}_d\xi^x$, $\dot{\delta}_k\xi^x$ because we do not have the explicit expression $x^{\alpha} = x^{\alpha}(u, v, \dot{u}, \dot{v})$ and $\dot{x}^{\alpha} = \dot{x}^{\alpha}(u, v, \dot{u}, \dot{v})$. This difficulty may be overcome in such a way that the mentioned expressions are substituted by anothers in which the partial derivatives with respect to x and $\dot{x}$ are present. Starting from (2.8) and (2.28) we may write

$$\text{(4.1)}\qquad \bar{D}\xi^a = B^a_{\alpha} D\xi^{\alpha} + (\theta^{\alpha}_b \xi^b + \nu^{\alpha}_m \xi^m) B^a_{\alpha} D$$

$$\text{(4.2)}\qquad \bar{D}\xi^n = N^n_{\alpha} D\xi^{\alpha} + (\theta^{\alpha}_b \xi^b + \nu^{\alpha}_m \xi^m) N^n_{\alpha} D$$

where

$$D = \lambda_c du^c + \lambda_k dv^k + (\mu_c + 2l_c)\bar{D}l^c + (\mu_k + 2l_k)\bar{D}l^k = 0.$$

On the other hand

$$\text{(4.3)}\qquad D\xi^{\alpha} = \delta_{\delta}\xi^{\alpha} dx^{\delta} + \dot{\delta}_l \xi^{\alpha} d\dot{x}^l + \overset{*}{\Gamma}{}^{\alpha}_{\beta\delta}\xi^{\beta} dx^{\delta} + A_{\beta}{}^{\alpha}{}_{\delta}\xi^{\beta} Dl^{\delta}.$$

Substituting

$$d\dot{x}^l = F(\delta^l_{\delta} - A_0{}^l{}_{\delta} Dl^{\delta} - \overset{*}{\Gamma}{}^l_{\delta} dx^{\delta} + F^{-1} dF \dot{x}^l$$

in (4.3) and using the notations

$$\text{(4.4)}\qquad \xi^x_{,\delta} = \delta_{\delta}\xi^x - \dot{\delta}_l \xi^x \overset{*}{\Gamma}{}^l_{\delta}$$

$$\text{(4.5)}\qquad \xi^x_{;\delta} = F\dot{\delta}_l \xi^x (\delta^l_{\delta} - A_0{}^l{}_{\delta})$$

$(x = b$ or $x = m)$ we obtain

$$\text{(4.6)}\quad \begin{aligned} D\xi^{\alpha} =& (\xi^b B^{\alpha}_{b|\delta} + \xi^m N^{\alpha}_{m|\delta} + B^{\alpha}_b \xi^b_{,\delta} + N^{\alpha}_m \xi^m_{,\delta})(B^{\delta}_c du^c + N^{\delta}_k dv^k) \\ &+ (\xi^b B^{\alpha}_b\big|_{\delta} + \xi^m N^{\alpha}_m\big|_{\delta} + B^{\alpha}_b \xi^b_{;\delta} + N^{\alpha}_m \xi^m_{;\delta})(B^{\delta}_c \bar{D}l^c + N^{\delta}_k \bar{D}l^k). \end{aligned}$$

Substituting (4.6) into (4.1) and (4.2), using the notations of (2.29) and (2.3) after a comparation with (3.14) and (3.15) we get

$$\text{(4.7)}\quad \begin{aligned} &\text{(a)} \qquad \xi^x_{\top c} = B^{\delta}_c \xi^x_{,\delta} + \overset{*}{\bar{\Gamma}}{}^x_{bc}\xi^b + \overset{*}{\bar{\Gamma}}{}^x_{mc}\xi^m \\ &\text{(b)} \qquad \xi^x_{\top k} = N^{\delta}_k \xi^x_{,\delta} + \overset{*}{\bar{\Gamma}}{}^x_{bc}\xi^b + \overset{*}{\bar{\Gamma}}{}^x_{mk}\xi^m \\ &\text{(c)} \qquad \xi^x\overline{\big|}_{c} = B^{\delta}_c \xi^x_{;\delta} + \bar{A}^x_{bc}\xi^b + \bar{A}^x_{mc}\xi^m \\ &\text{(d)} \qquad \xi^x\overline{\big|}_{k} = N^{\delta}_k \xi^x_{;\delta} + \bar{A}^x_{cb}\xi^b + \bar{A}^x_{mk}\xi^m \end{aligned}$$

$$(x = a \text{ or } x = m).$$

Comparling (3.12) with (4.7) we get

$$\begin{aligned}
&\text{(a)} && \delta_c\xi^x - F\dot{\delta}_d\xi^x\bar{\Gamma}^{*\,d}_{0c} - F\dot{\delta}_k\xi^x\bar{\Gamma}^{*\,k}_{0c} = B^\delta_c\xi^x_{,\delta}\\
&\text{(b)} && \delta_m\xi^x - F\dot{\delta}_d\xi^x\bar{\Gamma}^{*\,d}_{0m} - F\dot{\delta}_k\xi^x\bar{\Gamma}^{*\,k}_{0m} = N^\delta_m\xi^x_{,\delta}\\
&\text{(c)} && F\dot{\delta}_d\xi^x(\delta^d_c - \bar{A}^d_{0c}) - F\dot{\delta}_k\xi^x\bar{A}^k_{0c} = B^\delta_c\xi^x_{;\delta}\\
&\text{(d)} && -F\dot{\delta}_d\xi^x\bar{A}^d_{0m} + F\dot{\delta}_k\xi^x(\delta^k_m - \bar{A}^k_{0m}) = N^\delta_m\xi^x_{;\delta}
\end{aligned} \tag{4.8}$$

$$(x = a \text{ or } x = m).$$

The same formulae hold when in (4.8) ξ^x is substituted by g_{ab} or g_{nt}. For g_{ab} (4.8a) takes the form

$$\begin{aligned}
\delta_c g_{ab} - F\dot{\delta}_d g_{ab}\bar{\Gamma}^{*\,d}_{0c} - F\dot{\delta}_m g_{ab}\bar{\Gamma}^{*\,m}_{0c} &= B^\delta_c g_{ab,\delta}\\
&= B^\delta_c[\delta_\delta(g_{\alpha\beta}B^\alpha_a B^\beta_b) - \dot{\delta}_\iota(g_{\alpha\beta}B^\alpha_a B^\beta_b)\Gamma^{*\,\iota}_{\delta}]
\end{aligned} \tag{4.9}$$

where we have used (4.4) in which ξ^x is substituted by g_{ab}.

If we substitute (4.9), $\bar{\Gamma}^*_{abc}$ and $\bar{\Gamma}^*_{bac}$ defined by (2.29a) into

$$\begin{aligned}
g_{ab\top c} = \delta_c g_{ab} - F\dot{\delta}_d g_{ab}\bar{\Gamma}^{*\,d}_{0c} - F\dot{\delta}_m g_{ab}\bar{\Gamma}^{*\,m}_{0c}\\
- g_{db}\bar{\Gamma}^{*\,d}_{ac} - g_{ad}\bar{\Gamma}^{*\,d}_{bc}
\end{aligned}$$

we get

$$g_{ab\top c} = B^\gamma_c B^\alpha_a B^\beta_b g_{\alpha\beta|\gamma} - \lambda_c(\theta_{ab} + \theta_{ba}) = \lambda_c(g_{ab} - \hat{\theta}_{ab}) \tag{4.10}$$

where we put $\hat{\theta}_{ab} = \theta_{ab} + \theta_{ba}$. It is evident that (4.10) and (3.28a) are the same formulae, but from (4.10) follows that $g_{ab\top c} = \lambda_c g_{ab}$ when we choose such a connection coefficients $\bar{\Gamma}^*_{abc}$ in which $\theta_{ab} = 0$.

Similarly, using (3.12c), (4.5) and (2.29c) we obtain

$$\begin{aligned}
g_{ab}\overline{\big|_c} &= F\dot{\delta}_d g_{ab}(\delta^d_c - \bar{A}^d_{0c}) - F\dot{\delta}_m g_{ab}\bar{A}^m_{0c}\\
&\quad - g_{db}\bar{A}^d_{ac} - g_{ad}\bar{A}^d_{bc}\\
&= B^\delta_c F\dot{\delta}_\iota(g_{\alpha\beta}B^\alpha_a B^\beta_b)(\delta^\iota_\delta - A^\iota_{0\delta}) - g_{\alpha\beta}B^\beta_b B^\delta_c B^\alpha_a\big|_\delta\\
&\quad - g_{\beta\alpha}B^\alpha_a B^\delta_c B^\beta_b\big|_\delta - (\mu_c + 2l_c)(\theta_{ab} + \theta_{ba})\\
\Rightarrow g_{ab}\overline{\big|_c} &= B^{\alpha\beta\delta}_{abc}(g_{\alpha\beta}\big|_\delta - (\mu_c + 2l_c)(\theta_{ab} + \theta_{ba})\\
&= \mu_c g_{ab} - (\mu_c + 2l_c)\hat{\theta}_{ab}
\end{aligned} \tag{4.11}$$

(4.11) is the same as (3.30a). from (4.11) follows that $g_{ab}\overline{\big|_c} = \mu_c g_{ab}$ when we choose such connection coefficients $\bar{A}_{abc}$ in which $\theta_{ab} = 0$.

We have

Theorem 4.1. *If in the recurrent Finsler space the connection coefficients* $\bar{\Gamma}$ *and* $\bar{A}$ *defined by* (2.29) *and* (2.30) *in which* $\theta_{ab} = 0$ *and* $\nu_{nk} = 0$ *are used* $(\nu_{nk} + \nu_{kn} = \hat{\theta}_{nk})$ *then*

$$\begin{aligned} g_{ab\top c} &= \lambda_c g_{ab}, & g_{ab\top k} &= \lambda_k g_{ab}, \\ g_{nt\top c} &= \lambda_c g_{nt}, & g_{nt\top k} &= \lambda_k g_{nt}, \\ g_{ab\overline{|}c} &= \mu_c g_{ab}, & g_{ab\overline{|}k} &= \mu_k g_{ab}, \\ g_{nt\overline{|}c} &= \mu_c g_{nt}, & g_{nt\overline{|}k} &= \mu_k g_{nt}. \end{aligned}$$

The part of proof are (4.10) and (4.11). The other formulae can be obtained in the similar way.

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REFERENCES

[1] H. Rund, *The Differential Geometry of Finsler Spaces*, Springer-Verlag, 1959.

[2] A. Moór, *Über eine Übertragungstheorie der metrischen Linienelementraume mit recurrentem Grundtensor*, Tensor (N.S.) **29** (1975), 47–63.

[3] I. Čomić, *Subspaces of recurrent Finsler spaces*, Publ. Inst. Math. (N.S.) (Beograd) **33** (**47**) (1983), 41–48.

[4] I. Čomić, *Family of subspaces in Finsler space*, Tensor (N.S.) **37** (1982), 187–197.

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