

## MIXED NORM SPACES OF ANALYTIC AND HARMONIC FUNCTIONS, I

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**Abstract.** For an increasing absolutely continuous function  $\varphi : (0, 1) \rightarrow (0, +\infty)$  we define the spaces  $H(p, q, \varphi)$ ,  $p > 0$ , and  $h(p, q, \varphi)$ ,  $p \geq 1$ , (of analytic and harmonic functions  $f$  on the unit disc, respectively) by the requirement that the function  $r \rightarrow \varphi(1-r)M_p(r, f)$ ,  $0 < r < 1$ , belongs to  $L^q(\varphi'(1-r)dr/\varphi(1-r))$ . These spaces are generalizations of those considered by Shields and Williams [17, 18] and Mateljević and Pavlović [13]. If  $\varphi(2t) \leq C\varphi(t)$  we construct certain equivalent norms and use them to find the duals of  $H(p, q, \varphi)$  and  $h(p, q, \varphi)$ . In particular, we have an improvement of the main result of [18]. Our main tools are a theorem of Hardy and Littlewood on Cesaro means of power series and a new integrability theorem for power series with positive coefficients.

### 0. Introduction

Throughout the paper  $h(U)$  is the class of all complex-valued harmonic functions on the open unit disc  $U$ ,  $\varphi$  is an increasing absolutely continuous function on the interval  $(0, 1]$  with  $\varphi(0+) = 0$ , and  $0 < q \leq \infty$ . Let  $X$  be a quasi-normed space contained in  $h(U)$  such that for every  $f \in s(X)$  the function  $r \rightarrow \|f_r\|_X$ ,  $0 < r < 1$ , is measurable, where

$$s(X) = \{f \in h(U) : f_r \in X \text{ for all } r \in (0, 1)\},$$

while  $f_\xi$ ,  $|\xi| \leq 1$ , is defined by  $f_\xi(z) = f(\xi z)$ ,  $|z| < 1/|\xi|$ . We define

$$(0.1) \quad \|f\|_{X(q, \varphi)} = \left\{ \int_0^1 [\varphi(1-r)\|f_r\|_X]^q dm_\varphi(r) \right\}^{1/q}, \quad f \in s(X),$$

where

$$dm_\varphi(r)/dr = \varphi'(1-r)/\varphi(1-r), \quad 0 < r < 1.$$

We consider the spaces

$$X(q, \varphi) := \{f \in s(X) : \|f\|_{X(q, \varphi)} < \infty\}.$$

We shall mainly be concerned with the case when  $X$  is the usual Hardy space  $H^p$ ,  $0 < p \leq \infty$ . In this case we have  $s(X) = H(U)$  and  $\|f_r\|_X = M_p(r, f)$ , where  $H(U)$  stands for the subspace of  $h(U)$  consisting of analytic functions, and

$$M_p(r, f) = \left\{ \int_0^{2\pi} |f(re^{i\theta})|^p d\theta / 2\pi \right\}^{1/p}, \quad f \in h(U).$$

The space  $H^p(q, \varphi) =: H(p, q, \varphi)$ , is defined in [13] by using the measure  $dr / (1 - r)$  instead of  $dm_\varphi$ , but the present definition is more convenient for our purposes. Note that if  $\varphi(t) = t^\alpha$  for some  $\alpha > 0$ , then  $dm_\varphi(r) = \alpha dr / (1 - r)$ .

The spaces  $H(p, q, \alpha)$ , obtained by taking  $\varphi(t) = t^\alpha$ , have been considered by many authors. The spaces  $H(p, q, \alpha)$  are called Bergman spaces, although they were introduced by Džarbashian [3, 4]. However, many important theorems concerning  $H(p, q, \alpha)$  are contained in Hardy's and Littlewood works. See [5, 7] for information and references.

One of interesting problems is to describe the dual (and the predual if it exists) of  $H(p, q, \varphi)$ . The first result in this direction is due to Zakharyuta and Yudovich [19]. Using some bounded projections from  $L^\varphi(U)$  they found the dual of  $H(p, p, \alpha)$  with  $1 < p < \infty$ . In a similar way Shields and Williams [17] solved the duality problem for the space  $H(p, p, \varphi)$ ,  $1 \leq p \leq \infty$ , where  $\varphi$  is a "normal" function. Some other methods have been used by Duren, Romberg and Shields [6], Flett [7] and Shapiro [16]. However, these methods do not work in the case  $p < 1$ ,  $q > 1$  even if  $\varphi(t) = t^\alpha$ .

Recently Shields and Williams [18] considered the spaces  $h^\infty(\infty, \varphi)$ , where

$$h^p = \{f \in h(U) : \|f\|_p < \infty\}, \quad 0 < p \leq \infty,$$

$$\|f\|_p = \sup\{M_p(r, f) : 0 < r < 1\}, \quad f \in h(U).$$

They found the predual of  $h^\infty(\infty, \varphi)$  for such functions  $\varphi$  as  $\varphi(t) = 1/\log(e/t)$ ,  $0 < t \leq 1$ . In this case  $h^\infty(\infty, \varphi)$  is isomorphic to the dual of  $h^1(1, \varphi)$ , but this is not true if "h" is replaced by "H".

In this paper we present a new approach to the duality problem for  $H(p, q, \varphi)$  which can be used whenever  $0 < p, q < \infty$  and

$$(0.2) \quad \sup \varphi(2t)/\varphi(t) < \infty \quad (0 < t < 1/2).$$

If this condition is satisfied we say that  $\varphi$  is *quasi-normal*. If, in addition,

$$(0.3) \quad \sup \varphi(at)/\varphi(t) < 1 \quad (0 < t < 1)$$

for some  $a > 0$  then  $\varphi$  is said to be *normal*.

Our method is based on an extension of the following result of Mateljević and Pavlović [13].

**THEOREM A.** *Let  $\Delta_n(z) = \sum_{j \in J_n} z^j$ , where  $J_0 = \{0, 1\}$  and  $J_n = \{j : 2^n \leq j \leq 2^{n+1} - 1\}$  for  $n \geq 1$ . If  $1 < p < \infty$  and  $\varphi$  is normal then*

$$\left\{ \int_0^1 [\varphi(1-r)M_p(r, f)]^q dr / (1-r) \right\}^q \sim \left\{ \sum_{n=0}^{\infty} [\varphi(2^{-n})\|\Delta_n * f\|_p]^q \right\}^{1/q}, \quad f \in H(U).$$

This theorem does not hold if  $p \leq 1$  or if  $\varphi$  is not normal, and we shall use some polynomials more complicated than  $\Delta_n$ . Namely, for any lacunary sequence  $w = \{\lambda_n\}_0^\infty$  of positive integers and any integer  $N > 0$  we shall construct a sequence  $w_n = w_{n,N,\Lambda}$ ,  $n \geq 0$ , of harmonic polynomials satisfying the following conditions:

$$(0.4) \quad f = \sum_{n=0}^{\infty} w_n * f \quad \text{for all } f \in h(U),$$

with the series uniformly converging on compact subsets of  $U$ ;

$$(0.5) \quad \hat{w}_n(j) = 0 \quad \text{if } |j| \notin [\lambda_{n-1}, \lambda_n) \quad (\lambda_{-1} := 0);$$

$$(0.6) \quad \|w_n * f\|_X \leq C \|f\|_X, \quad f \in X, \quad n \geq 0$$

where  $X = H^p$ ,  $p > 1/(N+1)$ , and  $C$  is a positive real constant not depending on  $f$ ,  $n$ .

The construction of the sequence  $\{w_n\}$  will be given in Section 2. A theorem of Hardy and Littlewood [9] concerning Cesaro means of power series plays the central role in the proof of (0.6).

Our main result asserts that if  $\varphi(1/\lambda_n)$  behaves like  $2^{-n}$  then

$$(0.7) \quad \|f\|_{X(q,\varphi)} \sim \left\{ \sum_{n=0}^{\infty} [\varphi(1/\lambda_n) \|w_n * f\|_X]^q \right\}^{1/q}, \quad f \in s(X),$$

where  $X = H^p$ ,  $p > 1/(N+1)$ , or  $X = h^p$ ,  $p \geq 1$ . In fact, the conditions (0.6) and (0.7) are satisfied if  $X$  is an arbitrary Banach "A-space" (to be defined in Section 1). The proof of the main result is given in Sections 3, 4 and is based on a new integrability theorem for lacunary power series with positive coefficients.

Using the above properties of  $\{w_n\}$  one can reduce some problems for  $X(q, \varphi)$  to the analogous ones for  $X$ . In Section 5 we show how to calculate the dual of  $X(q, \varphi)$  if the dual of  $X$  is known. In particular, we have a solution to the duality problem for  $H(p, q, \varphi)$ ,  $p > 0$ , and for  $h(p, q, \varphi) := h^p(q, \varphi)$ ,  $p \geq 1$ , where  $\varphi$  is an arbitrary quasi-normal function. For example, it follows from our duality theorem and the Fefferman theorem that the dual of  $H(1, q, \varphi)$  is isomorphic to the space  $BMOA(q', \varphi)$ , where  $BMOA$  is the space of analytic functions of bounded mean oscillation. The dual of  $h(p, q, \varphi)$ ,  $p \geq 1$ , is isomorphic to  $h(p', q', \varphi)$ , and this generalizes Theorem 4 [18]. See also Section 6.

If  $\varphi$  is a normal function then the dual of  $H(p, q, \varphi)$  is simpler than in the general case (Section 7). If  $p \geq 1$  it is isomorphic to  $H(\infty, q', \varphi)$ . Furthermore, in this case the spaces  $H(p, q, \varphi)$  and  $H(1, q, \psi)$ , where,  $\psi(t) = \varphi(t)t^{1/p-1}$  have the same dual.

In Part II we shall consider the coefficient multipliers from  $X(q, \varphi)$  to  $Y(q, \psi)$ . For example, we shall prove that if  $\psi(t) = \chi(\varphi(t))$ , where  $\chi$  is normal, then  $H(p, q, \varphi)$  and  $H(p, q, \psi)$  are isomorphic via a multiplier transform; this generalizes the well-known theorem of Hardy and Littlewood on fractional integration

and differentiation in  $H(q, p, \alpha)$ . We shall also present a solution to a problem of Shields and Williams [18, Problem B].

### 1. A-Spaces

Let  $h(U_R)$  denote the class of all harmonic functions on the disc  $U_R = \{z : |z| < R\}$ ,  $R > 0$ . For a non-empty set  $E$  of integers let

$$h_E(U_R) = \{f \in h(U_R) : \text{supp}(\hat{f}) \subset E\},$$

where  $\hat{f}$  is uniquely determined by

$$f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n)r^{|n|}e^{in\theta}, \quad z = re^{i\theta} \in U_R.$$

A quasi-normed space  $X$  is said to be an  $A$ -space if there exists a set  $E$  such that the following conditions hold.

- I.  $X \subset h_E(U)$ , the inclusion being continuous.
- II.  $h_E(U_R) \subset X$  for all  $R > 1$ . More precisely, the restriction map  $f \rightarrow f|_U$  is a continuous operator from  $h_E(U_R)$  to  $X$ .
- III. If  $f \in X$  then  $f_\xi \in X$  and

$$\|f_\xi\| \leq \|f\| \text{ for } |\xi| \leq 1.$$

We suppose  $h_E(U_R)$  is endowed with the topology of uniform convergence on compact subsets of  $U_R$ . Recall that a quasi-norm  $\|\cdot\|$  on a linear space  $X$  is characterized by the following properties: 1.  $f = 0$  if  $\|f\| = 0$ ; 2.  $\|\lambda f\| = |\lambda| \cdot \|f\|$ ; 3.  $\|f + g\| \leq C\|f\| + C\|g\|$ , where  $C < \infty$  is independent of  $f, g \in X$ . Note also that if  $X$  satisfies Conditions I, II, then

$$s(X) = h_E(U).$$

It is hard to verify that  $H^p$  and  $h^p$ ,  $0 < p \leq \infty$ , are  $A$ -spaces. Many classical sequence spaces may be regarded as being  $A$ -spaces. For example, the space  $l^p$ ,  $0 < p \leq \infty$ , may be identified with the class of those  $f \in H(U)$  for which

$$\|f\|_p := \left\{ \sum_{n=0}^{\infty} |\hat{f}(n)|^p \right\}^{1/p} \leq \infty.$$

Further examples are Hardy-Orlicz spaces [11] and Orlicz sequence spaces.

Let  $\mathcal{P}$  be the set of all harmonic polynomials (i.e. of those  $f \in h(U)$  for which the support of  $\hat{f}$  is finite). It follows from II that  $\mathcal{P} \cap h_E(U) \subset X$ . We denote by  $X^\circ$  the closure  $\mathcal{P} \cap h_E(U)$  in  $X$ .

**PROPOSITION 1.1.** *For an  $A$ -space  $X$  the following assertions hold.*

- (a)  $X^\circ$  is an  $A$ -space and  $X(q, \varphi) = X^\circ(q, \varphi)$ .

- (b)  $X(q, \varphi)$  is a complete  $A$ -space.  
(c) If  $f \in X(q, \varphi)$ ,  $q < \infty$ , then  $\|f_\xi - f\|_{X(q, \varphi)} \rightarrow 0$  as  $\xi \rightarrow 1$  ( $|\xi| \leq 1$ ).  
(d) If  $q < \infty$  then  $X(q, \varphi)^\circ = X(q, \varphi)$ .

*Proof.* The proof of (a) is straightforward. It is also easy to see that  $Y := X(q, \varphi)$  is a quasi-normed space satisfying Conaition III. Let  $f \in h_E(U_R)$ ,  $R > 1$ . Then  $f \in X$  and  $\|f_r\|_X \leq \|f\|_X$  ( $0 < r < 1$ ). Using this and (0.1) we obtain  $\|f\|_Y \leq \|f\|_X q^{-1/q} \varphi(1)$  where  $q^{-1/q} := 1$  for  $q = \infty$ . This implies  $h_E(U_R) \subset Y$  with the continuous inclusion. On the other hand, if  $f \in Y$  and  $0 < \varrho < 1$  then  $f_\varrho \in h_E(U_{1/\varrho}) \subset X$ . Since  $\|f_r\|_X$ ,  $0 < r < 1$ , is a non-decreasing function (by III) we have

$$\|f\|_Y \geq \left\{ \int_\varrho^1 [\varphi(1-r)\|f_\varrho\|_X]^q dm_\varphi(r) \right\}^{1/q} = \|f_\varrho\|_X q^{-1/q} [\varphi(1) - \varphi(\varrho)].$$

This implies  $Y \subset h_E(U)$ .

To prove that  $Y$  is complete let  $\{f^n\}_0^\infty$  be a Cauchy sequence in  $X = X(q, \varphi)$ . In view of Conditions I, II and the completeness of  $h_E(U)$ , there exists  $f \in h_E(U)$  such that  $\|f^n - f_r\| \rightarrow 0$  as  $n \rightarrow \infty$ , for all  $r$ ,  $0 < r < 1$ . Using Fatou's lemma we obtain  $\|f^n - f\|_Y \leq \lim_n \inf \|f^m - f^n\|_Y$ , and this concludes the proof of (b).

For the proof of (c) observe that, whenever  $0 < r < 1$ ,  $f_{\xi_r}$  tends to  $f_r$  in the topology of  $h_E(U_{1/r})$  as  $\xi \rightarrow 1$  ( $|z| \leq 1$ ). Hence, by II,  $\|f_{\xi_r} - f_r\| \rightarrow 0$  as  $\xi \rightarrow 1$ . On the other hand,  $\|f_{\xi_r} - f_r\|_\lambda \leq C\|f_{\xi_r}\|_X + C\|f_r\|_X \leq 2C\|f_r\|_X$ , where  $C < \infty$  is independent of  $r$ . Now the required result follows from the Lebesgue dominated convergence theorem.

The assertion (d) is a consequence of (c) and the following proposition.

**PROPOSITION 1.2.** *Let  $f \in Y$ , where  $Y$  is an  $A$ -space. Then the following statements are equivalent: (i)  $f \in Y^\circ$ ; (ii)  $\|f_\xi - f\| \rightarrow 0$  as  $\xi \rightarrow 1$  ( $\xi \in U$ ); (iii)  $\|f_r - f\| \rightarrow 0$  as  $r \rightarrow 1^-$ .*

*Proof.* (iii)  $\Rightarrow$  (i). Let  $\varepsilon \rightarrow 0$  and choose  $r$  so that  $\|f_r - f\| < \varepsilon$ . Since  $f_r \in Y^\circ$  (Proposition 1.1 (a)) there exists  $g \in \mathcal{P} \cap Y$  such that  $\|g - f_r\| < \varepsilon$ . Hence  $\|f - g\| \leq C\|f - f_r\| + C\|f_r - g\| \leq 2C\varepsilon$ , where  $C$  is independent of  $\varepsilon$ .

(i)  $\Rightarrow$  (ii). It is easily verified that (ii) holds for  $f \in \mathcal{P}$ . Let  $f \in Y$ , let  $\varepsilon > 0$  and choose  $g \in \mathcal{P} \cap Y$  so that  $\|f - g\| < \varepsilon$ . Since  $\|g_\xi - f_\xi\| = \|(g - f)_\xi\| \leq \|g - f\|$  we get

$$\|f - f_\xi\| \leq 2C\varepsilon + C\|g - g_\xi\|, \quad |\xi| \leq 1.$$

Finally, we choose  $b > 0$  so that  $|\xi - 1| < \delta$  implies  $\|g - g_\xi\| < \varepsilon$ . This concludes the proof because the implication (ii)  $\Rightarrow$  (iii) is clear.

Next we consider some properties of dual spaces. The dual  $X'$  of a quasi-normed space  $X$  is the linear space of all bounded linear functionals on  $X$ , and is endowed with the norm

$$\|L\| = \sup\{|L(f)| : f \in X, \|f\|_X \leq 1\}.$$

Let  $X, Y \in h(U)$  be quasi-normed spaces. A function  $g \in h(U)$  is said to be a multiplier from  $X$  to  $Y$  if the map  $f \rightarrow f * g$  acts as a bounded linear operator from  $X$  to  $Y$ . If  $X, Y$  are  $A$ -spaces then

$$(X \rightarrow Y) := \{g \in s(X) \cap s(Y) : g \text{ is a multiplier from } X \text{ to } Y\}$$

is a quasi-normed space with the quasi-norm

$$\|g\|_{X,Y} = \sup\{\|f * g\|_Y : f \in X, \|f\|_X \leq 1\}.$$

Here  $f * g$  stands for the convolution of  $f, g \in h(U)$ :

$$f * g(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n)\hat{g}(n)r^{|n|e^{in\theta}}, \quad z = re^{i\theta} \in U.$$

For an  $A$ -space  $X$  let  $X^* = (X \rightarrow h^\infty)$ . Let  $h(\bar{U})$  be the subspace of  $h^\infty$  consisting of those functions which are continuous on the closed unit disc  $\bar{U}$ .

**PROPOSITION 1.3.** *Let  $X$  be a complete  $A$ -space such that  $X = X^\circ$ . Then the following assertions hold.*

- (a) *If  $L \in X'$  then there exists a unique function  $g \in (X \rightarrow h(\bar{U}))$  such that  $L(f) = f * g(1)$ ,  $f \in X$ .*
- (b)  *$X^* = (X \rightarrow h(\bar{U}))$ .*
- (c) *If  $L(f) = f * g(1)$ ,  $f \in X$ , where  $g \in (X \rightarrow h(\bar{U}))$ , then  $L \in X'$  and  $\|L\| = \|g\|_{X^*}$ .*

*Proof.* (a) Let  $s(X) = h_E(U)$  and define  $h_n \in X$  by  $\hat{h}_n = 0$  if  $n \notin E$ , and  $h_n(re^{i\theta}) = r^{|n|}e^{in\theta}$  if  $n \in E$ . If  $L \in X'$  define  $g \in h_E(U)$  by  $\hat{g}(n) = L(h_n)$ . Let  $\xi = \rho e^{it} \in U$ . Since  $f_\xi \in h_E(U_{1/|\xi|}) \subset X$  the series  $\sum_{-\infty}^{\infty} \hat{f}(n)h_n \rho^{|n|}e^{int}$  is convergent in  $X$ , whence  $f * g(\xi) = L(f_\xi)$ , for all  $f \in X$ . On the other hand, by Proposition 1.2 and the assumption  $X = X^\circ$ ,  $L(f_\xi) \rightarrow L(f_a)$  as  $\xi \rightarrow a$ , for all  $a \in \bar{U}$ . This implies  $f * g \in h(\bar{U})$  (for all  $f \in X$ ) and  $L(f) = f * g(1)$ . The function  $g$  belongs to  $(X \rightarrow h(\bar{U}))$  because  $|f * g(\xi)| = |L(f_\xi)| \leq \|L\| \|f_{|\xi|}\| = \|L\| \|f\|$  for  $\xi \in U$ . The uniqueness of  $g$  is obvious.

(b) Let  $g \in X^*$  and define  $L_\xi$ ,  $|\xi| < 1$ , by  $L_\xi(f) = f * g(\xi)$ ,  $f \in X$ . Then  $\{L_\xi : |\xi| < 1\}$  is a bounded subset of  $X'$ . On the other hand, if  $f \in \mathcal{P} \cap X$  and  $|a| = 1$  then the limit of  $L_\xi(f)$  as  $\xi \rightarrow a$  exists, because  $f * g \in \mathcal{P}$ . Since  $\mathcal{P} \cap X$  is dense in  $X$  and  $X$  is complete we conclude that the above limit exists for all  $f \in X$ . Therefore  $g$  belongs to  $(X \rightarrow h(\bar{U}))$ . The inclusion  $(X \rightarrow h(\bar{U})) \subset X^*$  is obvious.

The proof of (c) is straightforward.

Since  $(X^\circ)^* = X^*$  we have the following consequence of Proposition 1.3.

**PROPOSITION 1.4.** *If  $X$  is a Banach  $A$ -space then*

$$\|f\|_X = \sup\{\|f * g\|_\infty : g \in X^*, \|g\|_{X^*} \leq 1\}, \quad f \in X^\circ.$$

As an application we have

PROPOSITION 1.5. *Let  $X$  be a Banach  $A$ -space and let  $g$  be a harmonic polynomial. Then  $\|f * g\|_X \leq \|f\|_X \|g\|_1$   $f \in X$ .*

*Proof.* Since  $f * g$  is a polynomial we may apply Proposition 1.4 to obtain

$$\|f_r * g\|_X = \sup\{\|f_r * g * h\|_\infty : \|h\|_{X^*} \leq 1\}, \quad 0 < r < 1.$$

It is well-known and easy to see that  $\|(f_r * h) * g\|_\infty \leq \|f_r * h\|_\infty \|g\|_1$ . Hence  $\|f_r * g\|_X \leq \|f_r\|_X \|g\|_1 \leq \|f\|_X \|g\|_1$ . Now the desired result is verified because  $\|f_r * g\|_X = \|(f * g)_r\|_X \rightarrow \|f * g\|_X$  ( $r \rightarrow 1^-$ ).

## 2. On the sequence $\{w_n\}$

In what follows  $A = \{\lambda_n\}_0^\infty$  is a lacunary sequence of positive integers ( $\lambda_{n+1}/\lambda_n \geq c > 1$ ) and  $N$  is a positive integer. We shall construct a sequence  $W = \{w_n\}_0^\infty$  satisfying the conditions mentioned in Introduction. In fact, we can do somewhat more: this sequence satisfies

$$(2.1) \quad \left\| \sum_{j=0}^n w_j * f \right\|_X \leq C \|f\|_X, \quad f \in X, \quad n \geq 0.$$

The letter "C" denotes an arbitrary positive real constant which need not be the same on each occurrence.

It is clear that (2.1) implies (0.6). And it is a consequence of (0.4) and (2.1) that

$$\left\| f - \sum_{j=0}^n w_j * f \right\|_X \rightarrow 0 \quad (n \rightarrow \infty) \quad \text{for } f \in X^\circ.$$

THEOREM 2.1. *There exists a sequence  $W = \{w_n\}_0^\infty$  satisfying the conditions (0.4), (0.5) and (2.1) for all Banach  $A$ -spaces  $X$  and for all  $X \in \{H^p : p > 1/(N+1)\}$ .*

In the case of  $H^p$  spaces the proof depends on the following theorem of Hardy and Littlewood [9]. An elegant elementary proof is found by Colzani [2].

THEOREM HL. *If  $X = H^p$ ,  $p > 1/(N+1)$ , then*

$$(2.2) \quad \|\sigma_n^N f\|_X \leq C \|f\|_X, \quad f \in X, \quad n \geq 0.$$

Here  $\sigma_n^N f$ ,  $n = 0, 1, 2, \dots$ , are the  $(C, N)$  means of the series  $\sum \hat{f}(k) r^{|k|} e^{ik\theta}$ . They are defined for all  $f \in h(U)$  by  $\sigma_n^N f = K_n * K_{n+1} * \dots * K_{n+N-1} * f$ , where

$$K_n(re^{i\theta}) = \sum_{s=-n}^n \left(1 - \frac{|s|}{n+1}\right) r^{|s|} e^{is\theta}.$$

Using Proposition 1.5 and the well-known fact that  $\|K_n\|_1 = 1$  we obtain the following variant of Fejer's theorem.

THEOREM 2.2. *If  $X$  is a Banach  $A$ -space then the condition (2.2) with  $C = 1$  holds.*

For our purposes it is convenient to replace  $\sigma_n^N f$  by  $K_{n,N} * f$ , where

$$K_{n,N}(re^{i\theta}) = \sum_{s=-n}^n \left(1 - \frac{|s|}{n+1}\right)^N r^{|s|} e^{is\theta}.$$

LEMMA 2.1. *If an  $A$ -space  $X$  satisfies (2.2) then*

$$\|K_{n,N} * f\|_X \leq C \|f\|_X, \quad f \in X, \quad n \geq 0.$$

*Proof.* It is clear that there exist  $A_{j,n}$  ( $n \geq 0$ ,  $1 \leq j \leq N$ ) such that

$$\left(1 - \frac{t}{n+1}\right)^N = \sum_{j=1}^N A_{j,n} \prod_{s=1}^j \left(1 - \frac{t}{n+s}\right)$$

for all real  $t$ . Taking  $t = n+2, \dots, n+N$  we see that  $|A_{j,n}| \leq C(n+1)^{j-N}$ , where  $C$  depends only on  $N$ . This implies

$$\|K_{n,N} * f\|_X = \left\| \sum_{j=1}^N A_{j,n} \sigma_n^j f \right\|_X \leq C \sum_{j=1}^N (n+1)^{j-N} \|\sigma_n^j f\|_X, \quad f \in X,$$

where  $C$  depends only on  $N$ . This concludes the proof.

*Proof of Theorem 2.1.* For a fixed  $n \geq 0$  define the matrix  $B_n(t) = \{b_{mj}(t)\}$  ( $0 \leq m, j \leq N$ ) by

$$b_{mj}(t) = \begin{cases} \lambda_{n+m}^j, & 0 \leq j \leq N-1, \\ \lambda_{n+m}^N \xi_{n+m}(t), & j = N, \end{cases}$$

where

$$\xi_s(t) = \begin{cases} (1 - t/\lambda_s)^N, & 0 \leq t \leq \lambda_s \\ 0, & t \geq \lambda_s. \end{cases}$$

The function

$$\eta_n(t) := \det B_n(t) / \det B_n(0), \quad t \geq 0.$$

satisfies the condition

$$(2.3) \quad \eta_n(t) = 0 \quad \text{for } t \geq \lambda_{n+N}, \quad \text{and } \eta_n(t) = 1 \quad \text{for } 0 \leq t \leq \lambda_n.$$

The first equality holds because  $x_{n+m}(t) = 0$  for all  $t \geq \lambda_{n+N}$  and  $0 \leq m \leq N$ . To prove the rest of (2.3) observe that the function  $\eta_n$  coincides on the segment  $[0, \lambda_n]$  with a polynomial of degree  $\leq N$ . Thus, it suffices to show that  $d^k \eta_n(0) / dt^k = 0$  for  $1 \leq k \leq N$ . And this is immediate from the identity

$$d^k b_{mN}(0) / dt^k = k! \binom{N}{k} (-1)^k \lambda_{n+m}^{N-k}$$



by using the usual differentiation rule for determinants.

Now we define the sequence  $\{w_n\}$  by  $\hat{w}_0(k) = \eta_0(|k|)$  and

$$\hat{w}_n(k) = \eta_n(|k|) - \eta_{n-1}(|k|), \quad n \geq 1, \quad -\infty < k < \infty.$$

It follows from (2.3) that the condition (0.5) is satisfied. We also have

$$(2.4) \quad \sum_{n=1}^{\infty} \hat{w}_n(k) = 1, \quad -\infty < k < \infty.$$

If  $f$  is a harmonic polynomial then (0.4) holds because of (2.4) and the fact that  $w_n * f = 0$  for sufficiently large  $n$ . The general case of (0.4) is now obtained by using (2.1),  $X = h^\infty$ . It remains to verify the validity of (2.1).

LEMMA 2.2. *We have  $\eta_n(t) = \sum_{m=0}^N D_{n,m} \xi_{k+m}(t)$ ,  $t \geq 0$ , where:*  
 $\sup_{n,m} |D_{n,m}| \leq \infty$ .

*Proof.* For a finite sequence  $x_0, x_1, \dots, x_s$  let  $V(x_0, \dots, x_s)$  be the determinant of the matrix  $\{x_m^j\}_{m,j=0}^s$  (Vandermonde's determinant). It is easily seen that the desired identity holds with

$$|D_{n,0}| = \lambda_n^N V(\lambda_{n+1}, \dots, \lambda_{n+N})/V_n, \quad |D_{n,N}| = \lambda_{n+N}^N V(\lambda_n, \dots, \lambda_{n+N-1})/V_n,$$

and, for  $1 \leq m \leq N-1$ ,

$$|D_{n,m}| = \lambda_{n+m}^N V(\lambda_n, \dots, \lambda_{n+m+1}, \lambda_{n+m-1}, \dots, \lambda_{n+N})/V_n,$$

where  $V_n = V(\lambda_n, \dots, \lambda_{n+N})$ . Using the formula

$$V(x_0, \dots, x_s) = \prod_{j,m} (x_m - x_j) \quad (0 \leq j < m \leq s)$$

we find

$$|D_{n,m}|^{-1} = \lambda_{n+m}^{-N} \prod_{\substack{j=0 \\ j \neq m}}^N |\lambda_{n+j} - \lambda_{n+m}|, \quad 0 \leq m \leq N.$$

Since the sequence  $\{\lambda_n\}$  is lacunary there exists  $c > 0$  such that

$$|\lambda_{n+j} - \lambda_{n+m}| \geq c\lambda_{n+m}$$

for all  $j, m, n \geq 0$ ,  $j \neq m$ . This implies  $|D_{n,m}|^{-1} \geq c^N$ .

We return to the proof of Theorem 2.1. Since

$$\sum_{j=0}^n \hat{w}_j(k) = \eta_n(|k|) \quad \text{and} \quad \xi_{n+m}(|k|) = \hat{K}_{\lambda_{n+m-1}, N}(k)$$

we have

$$\sum_{j=0}^n w_j * f = \sum_{m=0}^N D_{n,m} K_{\lambda_{n+m-1}, N} * f.$$

Using Theorems HL and 2.2 and Lemmas 2.1 and 2.2 we obtain

$$\left\| \sum_{j=0}^n w_j * f \right\|_X \leq C \sum_{m=0}^N D_{n,m} \|f\|_X,$$

where  $C$  is independent of  $f \in X$  and  $n \geq 0$ . This concludes the proof of the theorem.

### 3. Quasi-normal functions

For the sake of convenience we suppose that all quasi-normal functions under discussion are defined, increasing and absolutely continuous on  $(0, \infty)$  and

$$(3.1) \quad \varphi(t)\varphi(1/t) \sim 1, \quad t > 0.$$

For non-negative functions  $F, G$  we write  $F(s) \sim G(s)$ ,  $s \in S$ , if there is  $C(0 < C < \infty)$  such that  $G(s)/C \leq F(s) \leq CG(s)$  for all  $s \in S$ . If the condition (0.2) is satisfied then

$$(3.2) \quad \varphi(at) \sim \varphi(t), \quad t > 0,$$

for all  $a > 0$ . Furthermore, there are  $C, \alpha > 0$  such that

$$(3.3) \quad \varphi(tu) \leq Cu^\alpha \varphi(t), \quad u > 1, \quad t > 0.$$

The following proposition is an immediate consequence of [10], Theorem II.1.1.

**PROPOSITION 3.1.** *If (3.2) holds then there exists a concave function  $\varphi_0$  on  $(0, \infty)$  such that  $\varphi(t) \sim \varphi_0(t)^\alpha$ ,  $t > 0$ .*

An increasing sequence  $\{A_n\}_{n=0}^\infty$  of positive real numbers is said to be *normal* if there are positive constants  $C, c$  such that

$$(3.4) \quad C^{-1}(1+c)^j \leq A_{n+j}/A_n \quad \text{and} \quad A_{n+1}/A_n \leq C, \quad n, j \geq 0.$$

This is equivalent to the requirement that  $1+c \leq A_{n+m}/A_n \leq C$ ,  $n \geq 0$ , for some  $C, c$  and some integer  $m > 0$ .

**PROPOSITION 3.2.** *Let  $\{A_n\}_0^\infty$  be a normal sequence and let  $\varphi$  be a quasi-normal function. Then there exists a lacunary sequence  $\{\lambda_n\}_0^\infty$  of positive integers such that  $\varphi(\lambda_n) \sim A_n$ ,  $n \geq 0$ .*

*Proof.* Let  $B_n = (1+c)^n \sup\{A_j(1+c)^{-j} : 0 \leq j \leq n\}$ ,  $n \geq 0$ , where  $c$  satisfies (3.4). Then  $B_n \sim A_n$ ,  $n \geq 0$ , and  $B_{n+1}/B_n \geq 1+c$ . Define  $t_n$  by  $\varphi_0(t_n) = B_n^{1/\alpha}$ ,  $n \geq 0$ , where  $\varphi_0$  satisfies the conclusion of Proposition 3.1. We have

$$\varphi_0(t_{n+1}) = \varphi_0(t_{n+1}t_n^{-1}t_n) \leq t_{n+1}t_n^{-1}\varphi_0(t_n)$$

because  $\varphi_0$  is concave,  $\varphi_0(0+) = 0$  and  $t_{n+1}/t_n > 1$ . Hence  $t_{n+1}/t_n > (1+c)^\alpha$  and consequently there is an integer  $n_0$  such that, for every  $n \geq n_0$ , the set  $[t_n, t_{n+1})$

contains at least two integers. We define  $\lambda_n$ ,  $n \geq 0$ , to be the smallest integer in  $[t_{n+n_0}, t_{n+n_0+1})$ . Since  $t_{n+n_0} \leq \lambda_n \leq 1 + t_{n+n_0}$  we have  $\lambda_{n+1}/\lambda_n \leq t_{n+n_0+1}/(1 + t_{n+n_0})$ , and this implies that  $\{\lambda_n\}$  is lacunary. The relation  $\varphi(\lambda_n) \sim A_n$  is obvious.

From now on we shall assume that  $A := \{\lambda_n\}$  is a lacunary sequence of positive integers such that the sequence  $\{\varphi(\lambda_n)\}$  is normal. (One can show that the existence of such a sequence is equivalent to the condition (0.2).)

**THEOREM 3.1.** *Let  $\{t_n\}_0^\infty$  be an increasing sequence of positive numbers such*

$$(3.5) \quad \varphi(t_{n+j})/\varphi(t_n) \geq C^{-1}(1+c)^j, \quad n, j \geq 0.$$

*Let  $F(r) = \varphi(1-r) \sup_{n \geq 0} a_n r^{t_n}$  or  $\varphi(1-r) \sum_{n=1}^\infty a_n r^{t_n}$ ,  $0 < r < 1$ , where  $a_n \geq 0$  for all  $n$ . Then*

$$(3.6) \quad C^{-1} \|\{\varphi(1/t_n)a_n\}\|_{l^q} \leq \|F\|_{L^q(dm_\varphi)} \leq C \|\{\varphi(1/t_n)a_n\}\|_{l^q},$$

*where  $C$  is independent of  $\{a_n\}$ .*

The case when  $\varphi$  is normal,  $t_n = 2^n$  and  $dm(r) = dr/(1-r)$  is discussed in [13]. Here we proceed in a similar way. Note that

$$dm_\varphi(r) = \varphi'(1-r)dr/\varphi(1-r).$$

**LEMMA 3.1.** *Let  $\{t_n\}$  be as in Theorem 3.1. and  $0 < s, \beta < \infty$ . Then*

$$\sum_{n=0}^\infty \varphi(t_n)^\beta r^{st_n} \leq C\varphi(1-r)^{-\beta}, \quad 0 < r < 1.$$

*Proof.* Since  $\varphi^\beta$  and  $\{st_n\}$  have the same properties as  $\varphi$  and  $\{t_n\}$  the lemma reduces to the case  $\beta = s = 1$ . First we prove that

$$(3.7) \quad \sum_{n=0}^\infty \varphi(\lambda_n)r^{\lambda_n-1} \leq C(1-r)^{-1}, \quad 0 < r < 1,$$

where  $\lambda_{-1} = 0$ . We may suppose  $r = 1 - 1/\lambda_m$  for some  $m$ . Then

$$\sum_{n=0}^\infty \varphi(\lambda_n)r^{\lambda_n-1} \leq \sum_{n=0}^m \varphi(\lambda_n) + \sum_{n=m+1}^\infty \varphi(\lambda_n)e^{-\lambda_n-1/\lambda_m}.$$

Since  $\{\varphi(\lambda_n)\}$  is normal we have  $\varphi(\lambda_n) \leq C(1+c)^{n-m}\varphi(\lambda_m)$ ,  $0 \leq n \leq m$ , and therefore

$$\sum_{n=0}^m \varphi(\lambda_n) \leq C\varphi(\lambda_m) = C\varphi(1/(1-r)) \leq C\varphi(1-r)^{-1}.$$

Similarly, using the inequalities  $\varphi(\lambda_n)/\varphi(\lambda_m) \leq C^{n-m}$  and  $\lambda_{n-1} \geq (1+c)^{n-m-1}\lambda_m$ ,  $n \geq m+1$ , we obtain

$$\begin{aligned} \sum_{n=m+1}^\infty \varphi(\lambda_n)e^{-\lambda_n-1/\lambda_m} &\leq \varphi(\lambda_m) \sum_{n=m+1}^\infty C^{n-m}e^{-(1+c)^{n-m-1}} \\ &= C\varphi(\lambda_m) \sum_{n=0}^\infty C^n e^{-(1+c)^n} \leq C\varphi(1-r)^{-1}. \end{aligned}$$

This completes the proof of (3.7).

Suppose now that  $\{t_n\}$  satisfies (3.5). Since  $\varphi(\lambda_{k+1})/\varphi(\lambda_k) \leq C$ ,  $k \geq 0$ , there exists  $j > 0$  such that

$$\varphi(t_{n+j})/\varphi(t_n) \geq \varphi(\lambda_{k+1})/\varphi(\lambda_k), \quad n, k \geq 0.$$

This implies that there is  $J > 0$  such that, for every  $k > 0$ , the set  $E_k := \{n \geq 0 : \lambda_{n-1} \leq t_n < \lambda_k\}$  contains at most  $J$  elements. (In fact,  $J = \max\{j, \text{card } E_0\}$ .) Hence

$$\sum_{n=0}^{\infty} (t_n)r^{t_n} = \sum_{k=0}^{\infty} \sum_{n \in E_k} \varphi(t_n)r^{t_n} \leq J \sum_{k=1}^{\infty} \varphi(\lambda_k)r^{\lambda_{k-1}} \leq C\varphi(1-r)^{-1}.$$

(If  $E_k = \emptyset$  we put  $\sum_{E_k} = 0$ .) This concludes the proof of Lemma 3.1.

LEMMA 3.2. *If  $\psi$  is a quasi-normal function then*

$$\int_0^1 \psi'(1-r)r^x dr \sim \psi(1/x), \quad x \geq 1.$$

*Proof.* It is proved in [13] that if  $\psi_0$  is normal then

$$\int_0^1 \psi_0(1-r)(1-r)^{-1}r^{x-1} dr \sim \psi_0(1/x), \quad x \geq 1.$$

The function  $\psi_0(t) := t\psi(t)$  is normal and therefore

$$\int_0^1 \psi'(1-r)r^x dr = x \int_0^1 \psi(1-r)r^{x-1} dr \sim x\psi_0(1/x) = \psi(1/x).$$

*Proof of Theorem 3.1.* Consider first the case  $q < \infty$ . Let  $s_n = t_n/2$ . Then, by means of Lemma 3.1,

$$\begin{aligned} F(r)/\varphi(1-r) &\leq \sum_{n=1}^{\infty} \varphi(t_n)^{1/2} r^{s_n} \\ &\leq \sup_{n \geq 0} a_n \varphi(t_n)^{-1/2} r^{s_n} \sum_{n=0}^{\infty} \varphi(t_n)^{1/2} r^{s_n} \\ &\leq C \sup_{n \geq 0} a_n \varphi(t_n)^{-1/2} r^{s_n} \varphi(1-r)^{-1/2}, \quad 0 < r < 1. \end{aligned}$$

Hence

$$F(r)^q \leq C\varphi(1-r)^{q/2} \sup_{n \geq 0} a_n^q \varphi(t_n)^{-q/2} r^{qs_n} \leq C\varphi(1-r)^{q/2} \sum_{n=1}^{\infty} a_n^q \varphi(t_n)^{-q/2} r^{qs_n}.$$

Integration yields

$$\int_0^1 F(r)^q dm_{\varphi}(r) \leq C \sum_{n=0}^{\infty} a_n^q \varphi(t_n)^{-q/2} \int_0^1 \psi'(1-r)r^{qs_n} dr.$$

where  $\psi(t) = \varphi(t)^{q/2}$ . Using Lemma 3.2, (3.2) and (3.1) we obtain the right hand side inequality in (3.6).

On the other hand, by Lemma 3.1,

$$\int_0^1 F(r)^q dm_\varphi(r) \geq C \sum_{n=1}^{\infty} \varphi(t_n) \int_0^1 F(r)^q \varphi'(1-r)r^{t_n} dr.$$

Hence, by the inequality  $F(r) \geq \varphi(1-r)a_n r^{t_n}$ ,

$$\int_0^1 F(r)^q dm_\psi(r) \geq C \sum_{n=0}^{\infty} \varphi(t_n) a_n^q \int_0^1 \psi_0'(1-r)r^{(q+1)t_n} dr,$$

where  $\psi_0(t) = \varphi(t)^{q+1}$ . The proof is finished using Lemma 3.2.

If  $q = \infty$  let  $A = \sup_{n \geq 0} a_n \varphi(1/t_n) < \infty$ . Then

$$F(r) \leq C \varphi(1-r) \sum_{n=1}^{\infty} \varphi(t_n) A r^{t_n} \leq CA,$$

where Lemma 3.1 has been used.

The rest follows from the inequality

$$F(1 - 1/t_n) \geq \varphi(1/t_n) a_n (1 - 1/t_n)^{t_n} \quad (t_n > 1).$$

As a consequence of Theorem 3.1 we have the following integrability theorem for power series with positive coefficients.

**THEOREM 3.2.** *Let  $J_0 = [0, \lambda_0)$  and  $J_n = [\lambda_{n+1}, \lambda_n)$ ,  $n \geq 1$ . The function  $F(r) = \varphi(1-r) \sum_{n=0}^{\infty} a_n r^n$ ,  $0 < r < 1$ , where  $a_n \geq 0$ , belongs to the space  $L^q(m_\varphi)$  if and only if*

$$\left\{ \varphi(1/\lambda_n) \sum_{J_n} a_k \right\}_{n=0}^{\infty} \in l^q.$$

The case when  $\varphi(t) = t^\alpha$  and  $\lambda_n = 2^n$  is known [12].

#### 4. Decomposition of $X(q, \varphi)$

The following theorem and Theorem 2.1 provide a sort of finite dimensional decomposition of  $X(q, \varphi)$ .

**THEOREM 4.1.** *Let  $X$  be a Banach  $A$ -space or  $X = H^p$ ,  $p > 0$ , let be  $\lambda$  a quasi-normal function, and let  $\{w_n\}_n^\infty$  satisfy (0.4), (0.5) and (0.6), where  $N \geq 0$  and the sequence  $\{\lambda_n\}_0^\infty$  is chosen so that  $\{\varphi(\lambda_n)\}_0^\infty$  is normal. Then (0.7) holds.*

We shall deduce Theorem 4.1 from Theorem 3.1. For  $g \in h(U)$  let  $g^j$  be defined by  $g^\circ(z) = g(O)$  and if  $j \geq 1$

$$\hat{g}^j(z) = \hat{g}(j)z^j + g(-j)\bar{z}^j.$$

Observe that  $g(z) = \sum_0^\infty g^j(z)$ ,  $z \in U$ .

LEMMA 4.1. *Let  $g = \sum_m^n g^j \in X$  ( $0 \leq m < n$ ), where  $X$  is as in Theorem 4.1. Then*

$$3^{-1}r^{2n}\|g\|_X \leq \|g_r\|_X \leq 2r^{m/2}\|g\|_X, \quad 0 < r < 1.$$

*Proof.* For the case  $X = H^p$  see [13]. Let  $X$  be a Banach space. After two summations by parts we find

$$g_r = \sum_0^\infty r^j g^j = (1-r)^2 \sum_0^\infty r^j (j+1) \sigma_j^1 g.$$

Taking into account that  $\sigma_j^1 g = 0$  for  $j < m$  and using the inequality  $\|\sigma_j^1 g\| \leq \|g\|$  (Theorem 2.2) we get

$$\|g_r\| \leq (1-r)^2 \sum_m^\infty r^j (j+1) \|g\| = r^m (1+m(1+r)).$$

Using the elementary inequality  $m(1-r) + 1 \leq 2r^{-m/2}$  we prove half of the lemma.

To prove the rest let  $R = 1/r > 1$  and  $f = g_r$ . Then two summations by parts give

$$g = \sum_0^n R^j f_j = \sum_0^{n-1} (R^j + R^{j+2} - 2R^{j+1})(j+1) \sigma_j^1 f + (R^n - R^{n+1})n \sigma_{n-1}^1 f + R^n f.$$

Hence, by Theorem 2.2,

$$\|g\| \leq (R-1)^2 \sum_0^{n-1} R^j (j+1) \|f\| + (R-1)R^n n \|f\| + R^n \|f\|.$$

Finally, we use the inequalities  $n(R-1) \leq R^n - 1 \leq R^n$  and

$$\sum_0^{n-1} R^j (j+1) \leq n \sum_0^{n-1} R^j = n(R^n - 1)(R-1)^{-1} nR \leq nR^n (R-1)^{-1}.$$

We obtain  $\|g\| \leq 3R^{2n} \|f\|$ , and this concludes the proof.

*Remark.* If  $X$  is a Banach space of analytic functions then

$$r^n \|g\| \leq \|g_r\| \leq r^m \|g\|.$$

This follows from the special case  $X = H^\infty$  [13] by using Proposition 1.4.

LEMMA 4.2. *Let  $X$  and  $\{w_n\}$  be as in Theorem 4.1. Let  $\beta = p$  if  $X = H^p$ ,  $p < 1$ , and  $\beta = 1$  if  $X$  is a Banach space. Then there exists a constant  $c > 0$  such that*

$$c \sup_{n \geq 0} \|w_n * f\|_X r^{2\lambda_n + N} \leq \|f_r\|_X \leq 2 \left\{ \sum_{n=0}^\infty \|w_n * f\|_X^\beta r^{\beta\lambda_n - 1/2} \right\}^{1/\beta}$$

for all  $f \in s(X)$  ( $= h_E(U)$  for some  $E$ ) and  $0 < r < 1$ .

Here we put  $\lambda_{n-1} = 0$ .

*Proof.* The first inequality follows from (0.6) and Lemma 4.1, because  $g := w_n * f$  is of the form

$$g = \sum_{\lambda_{n-1}}^{\lambda_{n+N}} g^j.$$

on the other hand, it follows from (0.4) and the triangle inequality for  $\|\cdot\|_X^\beta$  that

$$\|f_r\|_X^\beta \leq \sum_{n=0}^{\infty} \|w_n * f_r\|_X^\beta.$$

Applying Lemma 4.1 we conclude the proof.

*Proof of Theorem 4.1.* The part " $\|f\| \geq \dots$ " of (0.7) follows immediately from Lemma 4.2, Theorem 3.1 ( $t_n = 2\lambda_{n+N}$ ) and the relation  $\varphi(2/\lambda_{n+N}) \sim \varphi(1/\lambda_n)$ ,  $n \geq 0$ . To prove the rest we put  $s = q/\beta$  (where  $\beta$  is as in Lemma 4.2),  $a_n = \|w_n * f\|_X$ ,  $\psi(t)^\beta = \varphi(t)$  and

$$F(r) = \psi(1-r) \sum_{n=0}^{\infty} a_n r^{\beta\lambda_{n-1}/2}, \quad 0 < r < 1.$$

By Lemma 4.3

$$2^{-\beta} \|f\|_{X(q,\varphi)}^\beta \leq \|F\|_{L^s(m_\varphi)} = \beta^{-1/s} \|F\|_{L^s(m)_\psi}.$$

Now we desired result is easily deduced from Theorem 3.1 (with  $\psi$ ,  $s$  instead of  $\varphi$ ,  $q$ ).

As an application of Theorem 4.1 we have a generalization of Theorem A. Let  $\{S_n\}_0^\infty$  be the unique sequence  $\{w_n\}_0^\infty$  satisfying (0.4) and (0.5) with  $N = 0$ . We have

$$\hat{S}_n(j) = \begin{cases} 1, & \lambda_{n-1} \leq |j| < \lambda_n \\ 0, & \text{otherwise.} \end{cases}$$

**THEOREM 4.2.** *If  $1 < p < \infty$  and the sequence  $\{\varphi(\lambda_n)\}_0^\infty$  is normal then*

$$\|f\|_{h(p,q,\psi)} \sim \|\{\varphi(1/\lambda_n)\|S_n * f\|_p\|_{l^q}, \quad f \in h(U).$$

*Proof.* This follows from Theorem 4.1 and the well known Riesz theorem: If  $1 < p < \infty$  then there is  $C < \infty$  such that  $\|\sum_0^n g^j\|_p \leq C\|g\|_p$  for all  $g \in h^p$ ,  $n \geq 0$ .

The following proposition shows that Theorem 4.2 is actually a generalization of Theorem A.

**PROPOSITION 4.1.** *Let  $X$  be an  $A$ -space and let  $\varphi$  be a normal function. Then*

$$\|f\|_{X(q,\varphi)} \sim \left\{ \int_0^1 [\varphi(1-r)\|f_r\|_X]^q dr / (1-r) \right\}^{1/q}, \quad f \in s(X).$$

*Proof.* Only the case  $q < \infty$  requires a proof. Let  $f \in s(X)$  and  $\xi(r) = \|f_r\|_X^q$ ,  $0 < r < 1$ . Integration by parts shows that

$$q\|f\|_{X(q,\varphi)}^q = \int_0^1 \varphi(1-r)^q d\xi(r).$$

Since  $\xi$  is non-decreasing we see that if  $\psi(t) \sim \varphi(t)$  then  $\|f\|_{X(q,\psi)} \sim \|f\|_{X(q,\varphi)}$ . By Proposition 3.1 we may take  $\psi(t) = \psi_0(t)^\alpha$ , where  $\psi_0$  is a concave function. Then the function  $\psi_0(t)/t$ ,  $t > 0$ , is non-increasing and consequently  $t\psi_0'(t) \leq \psi_0(t)$  for almost all  $t > 0$ . On the other hand, from the concavity of  $\psi_0$  it follows that

$$\psi_0'(t)(ut - t) \geq \psi_0(ut) - \psi_0(t), \quad u > 1.$$

Using (0.3) we find  $b$  so that  $\psi_0(bt) \geq 2\psi_0(t)$ . Then  $\psi_0'(t)t(b-1) \geq \psi_0(t)$ , whence

$$c/(1-r) \leq \psi'(1-r)/\psi(1-r) \leq \alpha/(1-r), \quad 0 < r < 1,$$

where  $c = \alpha/(b-1)$ . This concludes the proof.

## 5. Duality theorems

Shields and Williams [18] found the predual of  $h(\infty, \infty, \varphi)$  for a quasi-normal function  $\varphi$  satisfying the following condition.

(SW) There exist a positive finite Borel measure  $\mu$  on  $[0, 1)$  and a constant  $C < \infty$  such that

$$\varphi(n+1)^{-1} = \int_0^1 r^{2n} d\mu(r)$$

and

$$(n+1)|\varphi(n) + \varphi(n+2) - 2\varphi(n+1)| \leq C(\varphi(n+1) - \varphi(n))$$

for all integers  $n \geq 0$ .

Our duality theorem does not depend on (SW).

It follows from the proof of Proposition 4.1 that if  $\psi \sim \varphi$  then  $X(q, \psi) = X(q, \varphi)$ . Thus we may suppose  $\varphi^{1/\alpha}$  is concave for some integer  $\alpha > 0$  (Proposition 3.1).

A consequence is that  $\varphi^{-1/\alpha}$  is convex, while this implies that

$$(5.1) \quad (1/\varphi)^2 \text{ is convex on } (1, \infty).$$

In fact, if  $\varphi$  is defined on  $(0, 1]$  we may extend it by

$$b/\varphi(t) = \int_0^1 r^t \varphi'(1-r) dr, \quad t > 1,$$

where  $b$  is chosen so that  $\varphi(1) = \varphi(1+)$ . Then  $\varphi$  is increasing and absolutely continuous on  $(0, \infty)$  and satisfies (5.1) and (3.1) (by Lemma 3.2).

For a quasinormal function  $\psi$  define  $D^\psi : h(U) \rightarrow h(U)$  by

$$(D^\psi f)^\wedge(n) = \psi(|n|+1)\hat{f}(n), \quad -\infty < n < \infty.$$



Note that  $D^\psi$  is an isomorphism of  $h(U)$  onto itself. For  $q \in (0, \infty]$  let

$$q' = \begin{cases} \infty & \text{if } q \leq 1, \\ q/(q-1) & \text{if } 1 < q < \infty, \\ 1 & \text{if } q = \infty. \end{cases}$$

**THEOREM 5.1.** *Let  $X$  be a Banach  $A$ -space or  $X = H^p$ ,  $p > 0$ , and let  $\varphi$  be a quasi-normal function satisfying (5.1). Then the operation  $D^{\varphi^2}$  acts as an isomorphism from  $X(q, \varphi)^*$  onto  $X^*(q', \varphi)$ .*

Note that if  $q < \infty$  then  $X(q, \varphi)^* = (X(q, \varphi) \rightarrow h(\bar{U}))$  and the dual of  $X(q, \varphi)$  is naturally identified with  $X(q, \varphi)^*$ . (See Propositions 1.1 (d) and 1.3).

Since  $(h)^* = h^{p'}$ ,  $1 \leq p \leq \infty$ , we have a solution to the duality problem for  $h(p, q, \varphi)$ .

**THEOREM 5.2.** *Let  $1 \leq p \leq \infty$  and let  $\varphi$  be as in Theorem 5.1. Then the space  $h(p, q, \varphi)^*$  is isomorphic to  $h(p', q', \varphi)$ , via the operator  $D^{\varphi^2}$ .*

The analogous result for  $H(p, q, \varphi)$  holds if  $1 < p < \infty$ . If  $p = 1$  we use Fefferman's result that  $(H^1)^* = BMOA$ , the space of analytic functions of bounded mean oscillation [8]. If  $0 < p < 1$  then  $(H^p)^*$  is equal (up to an equivalent renorming) to the space  $M^p$  of those  $f \in H(U)$  for which

$$(5.2) \quad \|f\|_{M^p} := \|D^{1/p}f\|_{H(\infty, \infty, 1)} = \sup_{0 < r < 1} (1-r)M_\infty(r, D^{1/p}f) < \infty$$

This is a result of Duren, Romberg and Shields [6, 7]. The operator  $D^s : h(U) \rightarrow h(U)$ ,  $-\infty < s < \infty$ , is defined by

$$(D^s f)^\wedge(k) = (|k| + 1)^s \hat{f}(k).$$

Thus we have the following.

**THEOREM 5.3.** *If  $\varphi$  is as in Theorem 5.1 then the operator  $D^{\psi^2}$  is an isomorphism of  $H(p, q, \varphi)^*$  onto  $Y(q', \varphi)$ , where: 1.  $Y = H^{p'}$  if  $1 < p < \infty$ , 2.  $Y = BMOA$  if  $p = 1$ , 3.  $Y = M^p$  if  $p < 1$ .*

For the proof of Theorem 5.1 we define the space  $l_s^q(X)$ ,  $-\infty < s < \infty$ , to be the class of all sequences  $F = \{f_n\}_0^\infty$  such that  $f_n \in X$ ,  $n \geq 0$ , and

$$\|F\|_{l_s^q(X)} := \left\{ \sum_{n=0}^{\infty} [2^{-ns} \|f_n\|_X]^q \right\}^{1/q} < \infty.$$

Let  $w = \{w_n\}_0^\infty$  be a sequence satisfying (0.4) and (0.5) for some lacunary  $\{\lambda_n\}_0^\infty$  and  $N \geq 0$ . For an  $A$ -space  $X$  we denote by  $W_s^q(X)$  the class of those  $f \in s(X)$  ( $= h_E(U)$  for some  $E$ ) for which

$$\|f\|_{W_s^q(X)} := \|\{w_n * f\}\|_{l_s^q(X)} < \infty.$$

If the condition (0.6) is satisfied then  $W_s^q(X)$  is an  $A$ -space. Moreover, Proposition 1.1 remains true if we replace  $X(q, \varphi)$  by  $W_s^q(X)$ . We omit the proof.

The following lemma will be used instead of the Hahn-Banach theorem. (The Hahn-Banach theorem does not hold for quasi-normed spaces.)

LEMMA 5.1. *Let  $\bar{l}_s^q(X)$  denote the subspace of  $l_s^q(X)$  consisting of all  $\{f_n\}$  such that  $f_n = 0$  for sufficiently large  $n$ . If (0.6) holds then the operator  $V$  defined by*

$$VF = \sum_{n=1}^{\infty} w_n * f_n, \quad F = \{f_n\} \in \bar{l}_s^q(X),$$

is a bounded linear operator from  $\bar{l}_s^q(X)$  to  $W_s^q(X)$ .

*Proof.* It follows from (0.5) that

$$(5.3) \quad w_n * w_j = 0 \quad \text{for } |j - n| \geq N + 1.$$

Applying now (0.4) we get

$$w_n * VF = \sum_{j=n-N}^{n+N} w_n * w_j * f_j, \quad n \geq 0,$$

where  $w_j = f_j = 0$  for  $j < 0$ . Hence

$$\|w_n * VF\|_X \leq K^{2N} \sum_{j=n-N}^{n+N} \|w_n * w_j * f_j\|_X,$$

where  $K$  satisfies  $\|f + g\|_X \leq K\|f\|_X + K\|g\|_X$ . Using (0.6) we get

$$\|w_n * VF\| \leq C \sum_{j=n-N}^{n+N} \|f_j\|_X, \quad n \geq 0.$$

Now the desired result is obtained by the use of the following lemma.

LEMMA 5.2. *Let  $m$  be a non-negative integer. Then the operator  $S$  defined on real sequences by*

$$Sx = \left\{ \sum_{j=n-m}^{n+m} \xi_j \right\}_{n=0}^{\infty}, \quad x = \{\xi_j\}_0^{\infty} \quad (\xi_j := 0 \text{ for } j < 0)$$

acts as a bounded operator from  $l_s^q(R)$  to itself, where  $R$  is the real line.

*Proof.* It is easily seen that the operators  $S_j$ ,  $0 \leq j \leq 2m$ , defined by  $S_j x = \{\xi_{j+n-m}\}_{n=0}^{\infty}$  map continuously  $l_s^q(R)$  into  $l_s^q(R)$ . The operator  $S$  has the same property because  $S = \sum_0^{2m} S_j$ .

THEOREM 5.4. *If (0.6) holds then  $W_s^q(X)^* = W_{-s}^{q'}(X^*)$  (with equivalent norms).*

*Proof.* Define the sequence  $\{P_n\}_0^{\infty}$  by

$$P_n = \sum_{j=n-N}^{n+N} w_j \quad (w_j = 0 \text{ for } j < 0).$$

It follows from (0.4) and (5.3) that  $P_n * w_n = w_n$  for  $n \geq 0$ . Hence

$$f * g = \sum_{n=0}^{\infty} w_n * f * g = \sum_{n=0}^{\infty} P_n * f * w_n * g$$

and consequently

$$\|f * g\|_{\infty} \leq \sum_{n=0}^{\infty} \|P_n * f\|_X \|w_n * f\|_{X^*},$$

where  $f \in W_s^q(X)$ ,  $g \in W_{-s}^{q'}(X^*)$ . Using Hölder's inequality we get

$$\|f * g\|_{\infty} \leq \|\{P_n * f\}\|_{l_{s^q}(X)} \|g\|_{W_{-s}^{q'}(X^*)}.$$

Since

$$\|P_n * f\|_X \leq C \sum_{j=n-N}^{n+N} \|w_j * f\|_X$$

we have, by Lemma 5.2 ( $\xi_j = \|w_j * f\|_X$ ),

$$\|\{P_n * f\}\|_{l_{s^q}(X)} \leq C \|f\|_{W_{s^q}(X)},$$

and this concludes the proof of the inclusion  $W_{-s}^{q'}(X^*) \subset W_s^q(X)^*$ .

To prove the converse let  $g \in W_s^q(X)^*$  and define the operator  $T$  on  $\bar{l}_s^q(X)$  by

$$TF = (VF) * g = \sum_{n=1}^{\infty} f_n * g_n,$$

where  $g_n = w_n * g$ . It follows from Lemma 5.1 that  $T$  is a bounded operator from  $\bar{l}_s^q(X)$  to  $h^{\infty}$  with

$$\|T\| \leq C \|g\|_{W_{s^q}(X)^*},$$

where  $C$  is independent of  $g$ . Now it suffices to prove that

$$\|T\| \geq \|\{g_n\}\|_{l_{-s}^{q'}(X^*)}$$

Let  $0 < \varepsilon < 1$  and, for every  $n \geq 0$ , choose  $f_n \in X$  so that  $\|f_n\|_X = 1$  and  $f_n * g_n(1) = \|f_n * g_n\|_{\infty} \geq \varepsilon \|g_n\|_{X^*}$ . If  $\{a_n\} \in \bar{l}_s^q(R)$  then  $\{a_n f_n\} \in \bar{l}_s^q(X)$ ,

$$\|\{a_n f_n\}\|_{l_{s^q}(X)} = \|\{a_n\}\|_{l_{s^q}(R)}$$

and

$$\|T\{a_n f_n\}\|_{\infty} \geq \varepsilon \sum_0^{\infty} \|g_n\| a_n.$$

This implies

$$\|T\| \geq \varepsilon \|\{\|g_n\|\}\|_{l_{-s}^{q'}(R)} = \varepsilon \|\{g_n\}\|_{l_{-s}^{q'}(X^*)}.$$

This concludes the proof.

*Proof of Theorem 5.1.* Choose  $\{\lambda_n\}$  so that  $\varphi(\lambda_n) \sim 2^n$ ,  $n \geq 0$ . Applying first Theorems 2.1 and 4.1 and then Theorem 5.4 we obtain  $X(q, \varphi)^* = W_{-1}^{q'}(X^*)$ . On the other hand,  $X^*$  is a Banach  $A$ -space so that

$$\|D^{\varphi^2} f\|_{X^*(q', \varphi)} \sim \|D^{\varphi^2} f\|_{W_{-1}^{q'}(X^*)}, \quad f \in s(X) = s(X^*),$$

by Theorem 4.1. Thus it remains to prove the following.

LEMMA 5.3. *Let  $Y$  be a Banach  $A$ -space, let  $1/\psi$  be convex on  $(1, \infty)$ , and let  $\psi(\lambda_n) \sim \psi(\lambda_{n+1})$ ,  $n \geq 0$ . Then*

$$\|w_n * D^\psi f\|_Y \sim \psi(\lambda_n) \|w_n * f\|_Y, \quad n \geq 0, f \in s(Y).$$

*Proof.* For fixed  $f$  and  $n \geq 1$  let  $g = w_n * f$ ,  $k = \lambda_{n-1}$ ,  $m = \lambda_{n+N}$ . Then

$$w_n * D^\psi f = \sum_{j=k}^m A_j g^j,$$

where  $A_j = \psi(j+1)$ , and the functions  $g^j$  are defined as in Section 4. We have

$$\sum_k^m A_j g^j = \sum_k^{m-1} (A_j + A_{j+2} - 2A_{j+1})(j+1)\sigma_j^1 g + (A_m - A_{m+1})m\sigma_{m-1}^1 g + A_m g.$$

Hence, by Theorem 2.2,

$$\left\| \sum_k^{m-1} A_j g^j \right\| \leq \sum_k^{m-1} |A_j + A_{j+2} - 2A_{j+1}|(j+1) \|g\| (A_{m+1} - A_m) m \|g\| + A_m \|g\|,$$

where  $\|\cdot\| = |\cdot|_Y$ . Letting  $a_j = 1/A_j$  we have

$$A_j + A_{j+2} - 2A_{j+1} = -A_j A_{j+2} (a_j + (a_{j+2} - 2a_{j+1})) + 2A_j (A_{j+2} - A_{j+1}) (a_j - a_{j+1}).$$

Since  $1/\psi$  is convex the function  $F_u(t) := (1/\psi(t) - 1/\psi(u))/(u-t)$ ,  $1 \leq t < u$ , is non-increasing. Therefore

$$a_j - a_{j+1} = F_{j+2}(j+1) \leq F_{j+2}(1+j/2) \leq 2/(j+2)\psi(1+j/2) \leq C a_j/(j+1).$$

On the other hand, we have  $a_j + a_{j+2} - 2a_{j+1} \geq 0$  because  $1/\psi$  is convex. It follows that

$$\begin{aligned} \sum_k^{m-1} |A_j + A_{j+2} - 2A_{j+1}|(j+1) &\leq A_{m-1} A_{m+1} \sum_k^{m-1} (a_j + a_{j+2} - 2a_{j+1})(j+1) \\ &+ C \sum_k^{m-1} A_j (A_{j+2} - A_{j+1}) a_j = A_{m-1} A_m ((a_k - a_{k+1})k - (a_m - a_{m+1})m + a_k - a_m) \\ &+ C(A_{m+1} - A_{k+1}) \leq A_{m-1} A_m (C a_k + a_k) + C A_{m+1} \leq C \psi(\lambda_n). \end{aligned}$$

In the last step we used the estimates  $A_{m+1} = \psi(\lambda_{n+N} + 1) \leq C \psi(\lambda_n)$ ,  $a_k = \psi(\lambda_{n-1})^{-1} \leq C \psi(\lambda_n)^{-1}$ . In the same way we get

$$m(A_{m+1} - A_m) = A_m A_{m+1} (a_m - a_{m+1}) m \leq C A_{m+1} A_m a_m = C A_{m+1} \leq C \psi(\lambda_n).$$

Thus  $\|\sum_k^m A_j g^j\| \leq C\psi(\lambda_n)\|g\|$ .

In the other direction, let  $h = \sum_k^m A_j g^j$ . Then  $g = \sum_k^m a_j h^j$ .

Now we have

$$\begin{aligned} \|g\| &\leq \sum (a_j + a_{j+2} - 2a_{j+1})(j+1)\|h\| + (a_m - a_{m+1})_m\|h\| + a_m\|h\| \\ &= ((a_k - a_{k+1})k - (a_m - a_{m+1})m + a_k - a_m)\|h\| + (a_m - a_{m+1})m\|h\| \\ &\quad + a_m\|h\| \leq C(a_k + a_m)\|h\| \leq C\psi(\lambda_n)^{-1}\|h\|. \end{aligned}$$

This completes the proof of Theorem 5.1.

## 6. More on the dual of $X(q, \varphi)$

For a function  $\varphi$  (not necessarily quasi-normal) define the measure  $M = M_\varphi$  on  $[0, 1)$  by

$$dM_\varphi(r) = \varphi(1-r)^2 dm_\varphi(r) = \varphi(1-r)\varphi'(1-r)dr.$$

For  $f, g \in h(U)$  let

$$(6.1) \quad (f, g) = \int_0^1 f_r * g_r(1) dM(r),$$

provided that the integral exists. For example, an application of Hölder's inequality shows that if  $f \in X(q, \varphi) =: Y$ ,  $q \geq 1$ , and  $g \in X^*(q', \varphi) =: Z$ , then the function  $r \mapsto f_r * g_r(1)$  belongs to  $L^1(M_\varphi)$  and

$$|(f, g)| \leq \|f\|_Y \|g\|_Z.$$

The analogous fact for  $q < 1$  holds as well, but with  $|(f, g)| \leq C\|f\|\|g\|$ . This shows that if  $g \in X^*(q', \varphi)$  then  $(\cdot, g)$  is a bounded linear functional on  $X(q, \varphi)$ . In some cases the converse holds too.

*Proof.* We have only to prove that if  $L \in X(q, \varphi)'$  then there is  $g \in X^*(q', \varphi)$  such that  $L(f) = (f, g)$  for all  $f \in X(q, \varphi)$ . We extend  $\varphi$  to  $(0, \infty)$  by

$$\varphi(t)^{-2} = c \int_0^1 r^{2(t-1)} dM(r), \quad t > 1,$$

where  $c$  is chosen so that  $\varphi(1+) = \varphi(1)$ . Applying Lemma 3.2 with  $\psi = \varphi^2$  we see that (3.1.) holds. The condition (5.1) is obviously satisfied. It follows that if  $L \in X(q, \varphi)'$ ,  $q < \infty$ , then there exists a unique  $h$  such that  $D^{\varphi^2} h \in X^*(q', \varphi)$  and  $L(f) = f * h(1)$ . (See Theorem 5.1 and Proposition 1.1 (d) and 1.3.) Letting  $g = D^{\varphi^2} h$  we have

$$\begin{aligned} L(f) = f * h(1) &= \sum_{n=-\infty}^{\infty} \hat{f}(n)\hat{g}(n) \int_0^1 r^{2|n|} dM(r) \\ &= \int_0^1 \sum_{n=-\infty}^{\infty} \hat{f}(n)\hat{g}(n)r^{2|n|} dM(r) = (f, g) \end{aligned}$$

for all harmonic polynomials  $f \in X(q, \varphi)$ . Since such polynomials are dense in  $X(q, \varphi)$ , we have  $L(f) = (f, g)$  for all  $f \in X(q, \varphi)$ . This completes the proof.

Let  $\eta$  be a positive finite Borel measure on  $[0, 1]$ , and let  $h^1(\eta)$  be the subspace of  $L^1(d\eta(r)d\theta/2\pi)$  consisting of harmonic functions. In [18] Shields and Williams proved that if a quasi-normal function  $\varphi$  satisfies the condition (SW) (mentioned at the beginning of Section 5) then there exists a measure  $\eta$  such that  $h(\infty, \infty, \varphi)$  is isomorphic to the dual of  $h^1(\eta)$  in the pairing (6.1) with  $dM(r) = \varphi(1-r)d\eta(r)$ . However, the measure  $d$  is not given in an explicit form. Theorem 6.1 shows that one can take  $d\eta(r) = \varphi'(1-r)dr$  without additional restrictions on  $\varphi$ . Moreover, our proofs show that any of the measures  $d\eta(r) = \varphi(1-r)^{\beta-1}\varphi'(1-r)dr$  with  $\beta > 0$  can be used.

It should be remarked that if  $f \in h(p, q, \varphi)$ ,  $p, q \geq 1$ , and  $g \in h(p', q', \varphi)$  then the function  $f(z)g(\bar{z})$ ,  $z \in U$ , belongs to  $L^1(\mu)$ , where the measure  $\mu$  on  $U$  is defined by

$$d\mu(re^{i\theta}) = dM(r)d\theta/2\pi.$$

The bilinear form (6.1) can be written as

$$(f, g) = \int_U \int f(z)g(\bar{z})d\mu(z)$$

As further application of Theorem 5.1 we prove a result concerning the space

$$X(0, \varphi) := \{f \in X(\infty, \varphi) : \lim_{r \rightarrow 1^-} \varphi(1-r)\|f_r\|_X = 0\}.$$

**THEOREM 6.2.** *Let  $X$  be a Banach  $A$ -space and let  $\varphi$  be a quasi-normal function. Then the second dual of  $X(0, \varphi)$  is isometrically isomorphic to  $X(\infty, \varphi)$ .*

In the case  $X = H^\infty$  a slightly more general result is proved by Rubel and Shields [15].

*Proof.* It is easily seen that  $Y := X(0, \varphi)$  is the closure of the harmonic polynomials in  $Z := X(\infty, \varphi)$ , and that  $Y^* = Z^*$ . By Proposition 1.3 the dual of  $Y$  is canonically isometric to  $Y^*$ . Since  $Z^*$  is isomorphic to  $X^*(1, \varphi)$  we see that the harmonic polynomials are dense in  $Y^*$  as well. This implies that the second dual of  $Y$  is isometric to the space  $(Y^*)^*$ . Thus it remains to prove that  $(Y^*)^* = Z$  with equality of the norms. To see this let

$$S(f) = \sup\{\|f_\xi\|_Y : |\xi| < 1\}$$

for  $f \in h_E(U) = s(Y) = s(X)$ . It is clear that  $Z = \{f \in h_E(U) : S(f) < \infty\}$  and  $\|f\|_Z = S(f)$ . On the other hand, if  $f \in h_E(U)$  and  $|\xi| < 1$  we have

$$\|f_\xi\|_{(Y^*)^*} = \sup\{\|f_\xi * g\|_\infty : \|g\|_{Y^*} \leq 1\} = \|f_\xi\|_Y,$$

where Proposition 1.4 (with  $Y$  instead of  $X$ ) has been used. This implies

$$S(f) = \sup\{\|f_\xi * g\|_\infty : \|g\|_{Y^*} \leq 1, |\xi| < 1\} = \sup\{\|f * g\|_\infty : \|g\|_{Y^*} \leq 1\},$$

and this concludes the proof.

### 7. On the dual of $H(p, q, \varphi)$ when $\varphi$ is normal

Theorem 5.3 may be simplified if  $\varphi$  is supposed to be normal. We may assume that for some  $\alpha > 0$  the function  $\varphi(t)/t^\alpha$ ,  $t > 0$ , is non-increasing (Proposition 3.1). Then the function  $\varphi_\gamma$ ,  $\gamma > \alpha$ , defined by

$$(7.1) \quad \varphi(t)\varphi_\gamma(t) = t^\gamma, \quad t > 0,$$

is normal.

**THEOREM 7.1.** *If  $p \leq 1$  and (7.1) holds, where  $\varphi$  and  $\varphi_\gamma$  are normal functions, then the operator  $D^{\gamma+1/p-1}$  acts as an isomorphism of  $H(p, q, \varphi)^*$  onto  $H(\infty, q', \varphi_\gamma)$ .*

Some special cases of this theorem have been discussed by Shields and Williams [17] ( $p = q = 1$ ) and Mateljević and Pavlović [14] ( $p = 1$ ,  $q \geq 1$ ). The case  $p < 1$ ,  $q > 1$ , is new even if  $\varphi(t) = t^\alpha$ . For further information see [1, 7].

*Proof.* Consider first the case  $p < 1$ . Let  $\{w_n\}$  be a sequence described by Theorem 2.1 with  $N > 1/p - 1$  and  $\lambda_n = 2^n$ . An obvious modification of Theorem 5.4 shows that the norm in  $H(p, q, \varphi)^*$  is equivalent to

$$\left\{ \sum_0^\infty [\varphi(2^{-n})^{-1} \|w_n * f\|_p^*]^{q'} \right\}^{1/q'},$$

where  $\|\cdot\|_p^*$  stands for the norm in  $(H^p)^*$ . (This also follows from Theorems 5.1 and 4.1 and Lemma 5.3). By the Duren-Romberg-Shields theorem we have

$$\|w_n * f\|_p^* \sim \sup_{0 < r < 1} (1-r)M_\infty(r, w_n * D^{1/p}f), \quad f \in H(U), \quad n \geq 0;$$

see (5.2). Using Lemma 4.1 we find

$$\|w_n * f\|_p^* \sim 2^{-n} \|w_n * D^{1/p}f\|_\infty.$$

Now Lemma 5.3 gives

$$\|w_n * D^s f\|_\infty \sim 2^{n\gamma} \|w_n * f\|_p^*,$$

where  $s = \gamma + 1/p - 1$ . Hence

$$\varphi_\gamma(2^{-n}) \|w_n * D^s f\|_\infty \sim \varphi(2^{-n})^{-1} \|w_n * f\|_p^*.$$

This implies that the norm in  $H(p, q, \varphi)^*$  is equivalent to

$$\left\{ \sum_0^\infty [\varphi_\gamma(2^{-n}) \|w_n * D^s f\|_\infty]^{q'} \right\}^{1/q'}.$$

Now the desired result follows from Theorem 4.1.

If  $p = 1$  we proceed in a similar way as in the case  $p < 1$ . We have only to prove that

$$\|w_n * f\|_1^* \sim \|w_n * f\|_\infty, \quad f \in H(U), \quad n \geq 0.$$

It is clear that  $\|w_n * f\|_1^* \leq \|w_n * f\|_\infty$ . Let  $g(z) = (1 - z)^{-2}$ . By the definition of  $\|\cdot\|_1^*$  we have

$$\|w_n * f * g_r\|_\infty \leq \|w_n * f\|_1^* \|g_r\|_1 = \|w_n * f\|_1^* * (1 - r^2)^{-1}.$$

Taking  $r = 1 - 2^{-n}$  and using Lemma 4.1 we get

$$\|w_n * D^1 f\|_\infty = \|w_n * f * g\|_\infty \leq C 2^n \|w_n * f\|_1^*.$$

Hence, by Lemma 5.3,  $\|w_n * f\|_\infty \leq C \|w_n * f\|_1^*$ . This completes the proof of the theorem.

It should be noted that if  $p < 1$ ,  $q \leq 1$  then the requirement that  $\varphi$  is normal is not necessary for the validity of Theorem 7.1. Namely, for any quasi-normal function  $\varphi$  such that  $\varphi(t)/t^\alpha$  is non-increasing the function  $\varphi_\gamma(t) := t^\gamma/\varphi(t)$  ( $\gamma > \alpha$ ) is normal, and we have the following.

**THEOREM 7.2.** *If  $\varphi$  is a quasi-normal function then the space  $H(p, q, \varphi)^*$ , where  $p < 1$  and  $q \leq 1$ , is isomorphic to  $H(\infty, \infty, \varphi_\gamma)$  via the operator  $D^{\gamma+1/p-1}$ .*

*Proof.* Let  $Z = H(p, q, \varphi)^*$ ,  $p < 1$ ,  $q \leq 1$  and  $\psi(t) = \varphi(t)^2 t^{1/p}$ . It follows from Theorem 5. that

$$\|f\|_Z \sim \sup\{\varphi(1-r)(1-s)M_\infty(rs, D^\psi f) : 0 < r, s < 1\}.$$

This easily gives

$$\|f\|_Z \sim \sup\{\varphi(1-r)(1-r)M_\infty(r, D^\psi f) : 0 < r < 1\}.$$

Since the function  $t\varphi(t)$ ,  $t > 0$ , is normal we have, by Theorem 4.1,

$$\|f\|_Z \sim \sup_n 2^{-n} \varphi(2^{-n}) \|w_n * D^\psi f\|_\infty,$$

where  $\{w_n\}$  is as in the proof of Theorem 7.1. Using now Lemma 5.3 we see that

$$\|f\|_Z \sim \sup_n \varphi_\gamma(2^{-n}) \|w_n * D^{\gamma+1/p-1}\|_\infty.$$

Theorem 4.1 concludes the proof.

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