PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 40 (54), 1986, pp. 85–98

ASYMPTOTIC PROPERTIES OF THE RADON TRANSFORM IN $\mathbf{R}^n$

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Abstract. The notion of regular variation in $\mathbf{R}^n$ introduced by Yakymiv [6] is used to study the asymptotic properties of the Radon and dual Radon transform of $\mathbf{R}^n$. As a corollary, an n-dimensional version of a theorem of Aljančić, Bojanić and Tomić [5] is proved. This corollary complements results of Ostrogorski [8].

1. Introduction. In [1], Radon proved that for sufficiently smooth functions $f(x)$ on $\mathbf{R}^3$, one could recover f from its integrals over planes. This result has been extended to $\mathbf{R}^n$, $n \geq 2$ and other manifolds (see e.g. Helgason [2]). Let $f : \mathbf{R}^n \to \mathbf{R}$ be measurable and integrable on hyperplanes $x \cdot \omega = p$, where $\omega \in S^{n-1}$ (the n-sphere) and $p \in \mathbf{R}$. The Radon transform f is defined on $S^{n-1} \times \mathbf{R}$ by

$$\check{f}(\omega, p) = \int_{x\cdot\omega=p} f(x)dm(x), \tag{1.1}$$

where dm denotes Lebesgue measure on the hyperplane. Inversion of the Radon transform is achieved for Schwartz space functions via the following formula [2]:

$$f(x) = cL_x^{(n-1)/2} \int_{S^{n-1}} \check{f}(\omega, x \cdot \omega)d\omega, \tag{1.2}$$

where c is a constant dependant on n, L_x is the Laplacian on $\mathbf{R}^n$ and $d\omega$ denotes the surface element on S^{n-1}. Notice that when n is even, (1.2) involves a fractional power of the Laplacian defined using Riesz potentials [3].

The above formula suggests the dual transform: let g be defined on $S^{n-1} \times \mathbf{R}$, the dual tranform of g (provided it exists) is given by:

$$\breve{g}(x) = \int_{S^{n-1}} g(\omega, x \cdot \omega)d\omega. \tag{1.3}$$

See Helgason [2] for the basic properties of these transforms.

AMS Subject Classification (1980): Primary 44A15, 26B35.

The notion of regular variation in the sense of Karamata [4] is well known in Abelian and Tauberian theorems for various integral transforms, in particular one dimensional Fourier cosine and sine transforms [5, Lemma 2.3]. A radial notion of regular variation in $\mathbf{R}^n$ (see section 2) was defined by Yakymiv [6] and has been used by Ostrogorski [7], [8] to study asymptotic properties of integral transforms and Fourier transforms in $\mathbf{R}^n$.

In Section 3 we develop Abelian results for the Radon transform on $\mathbf{R}^n$ and in cones in $\mathbf{R}^n$. A modification of the well known relation between Radon and the n-dimensional Fourier transform $\tilde{f}$,

$$\tilde{f}(s\omega) = \int_R e^{-\mathrm{isp}} \hat{f}(\omega, p)\,dp, \tag{1.4}$$

allows application of our result in combination with one dimensional results [5] to obtain an Abelian result for the Fourier sine transform on cones in $\mathbf{R}^n$. This complements the result of Ostrogorski [8]. Section 4 develops the dual result whose main consequence is a Tauberian theorem.

2. Preliminaries. Regularly varying functions at infinity [4] on $(0, +\infty)$ are those positive measurable functions $r(t)$ such that for some $\alpha \in \mathbf{R}$,

$$\lim_{\lambda \to +\infty} \frac{r(\lambda t)}{r(\lambda)} = t^\alpha. \tag{2.1}$$

Furthermore these functions have been characterized to have the form $r(t) = t^\alpha l(t)$, where l is slowly varying at infinity i.e. for $t > 0$,

$$\lim_{\lambda \to +\infty} \frac{l(\lambda t)}{l(\lambda)} = 1. \tag{2.2}$$

Both limits (2.1) and (2.2) are known to be uniform for t contained in a finite closed interval; α is called the index of regular variation.

A notion of regular variation in cones in $\mathbf{R}^n$ was defined by Yakymiv [7] although a considerably stronger definition was given earlier [9]. We state a weakened version of Yakymiv's definition for the case of the cone being all of $\mathbf{R}^n$. A positive measurable function $R(x)$ is said to be regularly varying (at infinity) in $\mathbf{R}^n$ if there exists a $\varphi : \mathbf{R}^n \to \mathbf{R}_+$ and a fixed $e \in S^{n-1}$ such that

$$0 < \inf_{x \in S^{n-1}} \varphi(x), \quad \sup_{x \in S^{n-1}} \varphi(x) < +\infty \text{ and,}$$

$$\lim_{\lambda \to +\infty} \sup_{x \in B} \left| \frac{R(\lambda x)}{R(\lambda e)} - \varphi(x) \right| = 0 \tag{2.3}$$

for all compact sets $B \subset \mathbf{R}^n - \{0\}$. As noted in [6], the assumption of the uniform limit must be made in $\mathbf{R}^n$ as opposed to the one-dimensional case. Also, the class of regularly varying functions is independent of e. Furthermore, it follows that $R \in RV$ if and only if for some $\varphi : \mathbf{R}^n \to \mathbf{R}_+$ and some regularly varying function r on $(0, +\infty)$ we have,

$$\lim_{\lambda \to +\infty} \sup_{x \in B} \left| \frac{R(\lambda x)}{r(\lambda)} - \varphi(x) \right| = 0 \tag{2.4}$$

for all compact subsets $B \subset \mathbf{R}^n - \{0\}$. The function φ is homogeneous of order α, equal to the index of r.

If $\varphi \equiv 1$, then f is said to be slowly varying at infinity. Every regularly varying function f can be represented $f(x) = L(x)\varphi(x)$, where L is slowly varying. The number α is called the index of R, φ is the index function of R. Henceforth the class of regularly varying functions with index α is denoted $RV_\alpha(\mathbf{R}^n)$.

Let L be slowly varying at infinity in $\mathbf{R}^n$, then (2.4) becomes

$$\lim_{\lambda\to+\infty} \sup_{x\in B} \left| \frac{L(\lambda x)}{l(\lambda)} - 1 \right| = 0 \tag{2.5}$$

where $l(\cdot)$ is slowly varying on $(0,+\infty)$ and $B \in \mathbf{R}^n - \{0\}$ is compact. Writing $x = s\omega$ where $s > 0$ and $\omega \in S^{n-1}$, then (2.5) implies

$$\lim_{\lambda\to+\infty} \sup_{\omega\in S^{n-1}} \left| \frac{L(\lambda s\omega)}{l(\lambda)} - 1 \right| = 0 \tag{2.6}$$

In fact, the converse holds as follows; this is very useful later.

Lemma 2.1. *Let $L : \mathbf{R}^n \to \mathbf{R}_+$ be measurable. Then L is slowly varying at infinity if and only if* (2.6) *holds.*

Proof. We need show that (2.6) implies (2.5). If (2.5) does not hold, then for some $\varepsilon > 0$, there is a compact set $B \subset \mathbf{R}^n - \{0\}$, a sequence $\{x_n\} \subset B$ and a sequence $\{\lambda_n\} \subset \mathbf{R}_+$ with $\lambda_n \to +\infty$ such that

$$\left| \frac{L(\lambda_n x_n)}{l(\lambda_n)} - 1 \right| \geq \varepsilon, \quad n = 1, 2, \dots$$

Without loss of generality we may assume $x_n \to x (n \to +\infty)$. Let $x_n = r_n\omega^n$ where $\{\omega_n\} \subset S^{n-1}$, then,

$$\left| \frac{L((\lambda_n r_n)\omega_n)}{l(\lambda_n r)} - 1 \right| \geq \frac{l(\lambda n)}{l(\lambda_n r_n)} \left| \left| \frac{L(\lambda_n x_n)}{l(\lambda_n)} - 1 \right| - \left| \frac{l(\lambda_n r_n)}{l(\lambda_n)} - 1 \right| \right|.$$

As $\lim_{n\to+\infty} |l(\lambda_n r_n)/l(\lambda_n) - 1| = 0$, for sufficiently large n, we have,

$$\left| \frac{L((\lambda_n r_n)\omega_n)}{l(\lambda_n r_n)} - 1 \right| \geq \left(1 - \frac{\varepsilon}{2}\right) \frac{\varepsilon}{2}.$$

This contradicts (2.6) and concludes the proof.

Lemma 2.2. *Let L be slowly varying on $\mathbf{R}^n$. Then there exists a constant $a \in \mathbf{R}_+$ and measurable functions $\eta(x)$ on $|x| \geq a$, $\varepsilon(t)$ on $t \geq a$ such that*

$$L(x) = \exp\left\{ \eta(x) + \int_a^{|x|} \frac{\varepsilon(t)}{t} dt \right\}.$$

Also, $\lim_{|x|\to\infty} \eta(x) = c \in \mathbf{R}$ *and* $\lim_{t\to\infty} \varepsilon(t) = 0$.

Proof. For $e \in S^{n-1}$, define $l_r(|x|) = L(|x|e)$; then $\lim_{|x|\to\infty} \frac{L(x)}{l_r(|x|)} = 1$, i.e. l_r is a radial asymptotic equivalent of L. Also l_r is slowly varying on $(0, \infty)$; by the one-dimensional representation theorem [9] there are measurable functions $\eta_1(t)$, $\varepsilon(t)$ on $t \geq a$ such that

$$l(|x|) = \exp\left\{\eta_1(|x|) + \int_a^{|x|} \frac{\varepsilon(t)}{t} dt\right\}$$

and $\eta_1(|x|) \to c_1 \in \mathbf{R}$, $\varepsilon(t) = o(1)(t \to +\infty)$. Set $\eta(x) = \lg \frac{L(x)}{l_r(|x|)} + \eta_1(|x|)$ and the result follows.

Lemma 2.3. *Let L be slowly varying on $\mathbf{R}^n$ and let $\beta > 0$. Then there exist $M = M(\alpha)$ and $B = B(\alpha)$ such that*

$$\left|\frac{L(\lambda x)}{L(\lambda e)} - 1\right| \leq M|x|^\beta$$

for $\lambda \geq B$ and $|x| \geq 1$.

Proof. The proof is like that for the one dimensional case, we include it for completeness. For $\lambda \geq a$ and $|x| \geq 1$, we have

$$\frac{L(\lambda x)}{L(\lambda e)} = \exp\left\{\eta(\lambda x) - \eta(\lambda e) + \int_\lambda^{\lambda|x|} \frac{\varepsilon(t)}{t} dt\right\}.$$

Hence

$$\frac{L(\lambda x)}{L(\lambda e)} - 1 = \exp\left\{\eta(\lambda x) - \eta(\lambda e) + \int_\lambda^{\lambda|x|} \frac{\varepsilon(t)}{t} dt\right\} - 1.$$

Let $Q(x, \lambda, e) = \eta(\lambda x) - \eta(\lambda e) + \int_\lambda^{\lambda|x|} \frac{\varepsilon(t)}{t} dt$, then

$$\left|\frac{L(\lambda x)}{L(\lambda e)} - 1\right| \leq |Q(\lambda, x, e)| \exp|Q(\lambda, x, e)|.$$

Now, there is a constant $B \geq a$ such that if $|x| > 1$, $\lambda > B$, we have $|\eta(\lambda x) - c| < \beta/2$ and $|\varepsilon(t)| < \beta/2$. Hence

$$|Q(\lambda, x, e)| \leq \delta + \beta/2 \lg|x|.$$

and

$$|L(\lambda x)/L(\lambda e) - 1| \leq (\beta + \beta/2 \lg|x|) e^\beta |x|^{\beta/2}.$$

For $|x| \geq 1$, $\lg|x| \leq 2/\beta |x|^{\beta/2}$ so that

$$|L(\lambda x)/L(\lambda e) - 1| \leq \beta e^\beta (1 + 1/\beta|x|^{\beta/2} \leq \beta e^\beta (1 + 1/\beta)|x|^\beta.$$

This completes the proof with $M(\beta) = \beta e^\beta (1 + 1/\beta)$.

We quote a one-dimension result concerning the Fourier sine transform for later use.

LEMMA 2.4. (*Aljančić, Bojanić, Tomić* [5]). *Let* $-2 < \gamma < -1$ *and suppose* $L(t)$ *is slowly varying on* $(0, +\infty)$ *such that* $t^\delta L(t)$ *is bounded in every interval* $[0, \Delta]$ *for some* $\delta > 0$. *Then as* $y \to 0^+$,

$$\int_0^{+\infty} t^\gamma L(t) \sin ytdt \equiv \Gamma(1-\gamma) \cos \gamma\pi/2 \cdot y^{\gamma-1} L(1/y).$$

Remark. This says that the Fourier sine transform of $t^\gamma L(t)$ is regularly varying at zero with index $\gamma - 1$. In the n-dimensional case, a positive measurable function $R(x)$ is said to be regularly varying at zero if for some $e \in S^{n-1}$, some function $\varphi : \mathbf{R}^n \to \mathbf{R}_+$, and some $r \in \mathbf{R}V_\alpha(\mathbf{R}_+)$,

$$\lim_{\lambda \to +\infty} \sup_{x \in B} \left| \frac{R(x/\lambda)}{r(\lambda)} - \varphi(x) \right| = 0,$$

for every compact subset $B \subset \mathbf{R}^n - \{0\}$.

Finally, we introduce a notion of regular variation for even functions defined on $S^{n-1} \times \mathbf{R}$. This serves two purposes: succinct formulation of later results, and to bring out the familiar duality [2] (as homogeneous spaces) between $\mathbf{R}^n$ and the manifold $\mathbf{P}^n$, of hyperplanes in $\mathbf{R}^n$, in the realm of Abelian and Tauberian theorems. Note that $S^{n-1} \times \mathbf{R}$ is naturally identified with $\mathbf{P}^n$ via a double covering map, hence the restriction to even functions.

A positive even measurable function $g : S^{n-1} \times \mathbf{R} \to \mathbf{R}_+$ is said to be regularly varying (at infinity) if (i) for each $p \neq 0$, $g(\omega, p)$ is bounded in ω; (ii) for some regularly varying function $r(\lambda)$ on $(0, +\infty)$, and some $\varphi : S^{n-1} \times \mathbf{R} \to \mathbf{R}_+$,

$$\lim_{\lambda \to +\infty} \sup_{\omega \in S^{n-1}} \left| \frac{g(\omega, \lambda p)}{r(\lambda)} - \varphi(\omega, p) \right| = 0. \tag{2.5}$$

It should be noted that condition (i) is a technical necessity (see Section 4), its counterpart for $\mathbf{R}^n$ imposes no further restriction.

Several properties are almost transparent, we list them for the sake of brevity. The function φ is even and for each $\omega \in S^{n-1}$ is homogeneous in p for order a equal to the index of r. Also, φ is bounded on compact subsets of $S^{n-1} \times (\mathbf{R} - \{0\})$. The class of regularly varying functions on $S^{n-1} \times \mathbf{R}$ is denoted $RV_\alpha(S^{n-1} \times \mathbf{R})$. Finally, every $g \in RV_\alpha(S^{n-1} \times \mathbf{R})$ admits a presentation $g(\omega, p) = l(\omega, p)\varphi(\omega, p)$ where l is slowly varying on $S^{n-1} \times \mathbf{R}$ (with the obvious definitions).

3. Estimates of Radon and Fourier Transforms. The radial nature of the definition of regular variation (2.1) will be seen as natural due to the utility of polar representation. This is brought to light in the proof of the following result.

THEOREM 3.1. *Let* $\alpha < 1 - n$. *If* $f \in RV_\alpha(\mathbf{R}^n)$, *then* $\hat{f}(\omega, p)$ *is defined for* $\omega \in S^{n-1}$, $p > 0$ *and* $\hat{f} \in RV_\delta(S^{n-1} \times \mathbf{R})$ *where* $\delta = \alpha + n - 1$.

Proof. The proof given below is for $n \geq 3$, symbolic modification is needed for $n = 2$. We need prove that for $f(x) = \varphi(x)L(x)$, where L is slowly varying on

$\mathbf{R}^n$ and φ is the index funciton of f,

$$\lim_{\lambda\to+\infty} \frac{\hat{f}(\omega,\lambda p)}{\lambda^{\alpha+n-1} l(\lambda)\hat{\varphi}(\omega,1)} = p^{\alpha+n-1},$$

uniformly in $\omega \in S^{n-1}$, $l(\cdot)$ being the slowly varying function given in in (2.5). The existence of $\hat{f}$ is based on similiar methods and left to the reader. The idea of the proof is to express the integral (1.1) in polar form. For $\omega \in S^{n-1}$, let $\omega^{\perp}$ denote the orthogonal complement and set $S_\omega = \omega^{\perp} \cap S^{n-1}$. Note that S_ω has a natural measure and this measure can be nomalized so that the measure of S_ω is the surface measure of S^{n-2}, the value denoted ω_{n-2}. Now,

$$\hat{f}(\omega,p) = \int_{x\cdot\omega=p} f(x)dm(x) = \int_{S_\omega}\int_0^{+\infty} f(p\omega+s\omega')s^{n-2}dsd\omega',$$

where $d\omega'$ denotes measure on S_ω. A change of variables gives

$$\hat{f}(\omega,p) = p^{n-1}\int_{S_\omega}\int_0^{+\infty} f(p(\omega+s\omega'))s^{n-2}dsd\omega'.$$

For $f(x) = \varphi(x)L(x)$, we obtain for $p>0$:

$$\begin{aligned}&\frac{\hat{f}(\omega,\lambda p)}{\lambda^{\alpha+n-1}l(\lambda)} - p^{\alpha+n-1}\hat{\varphi}(\omega,1)\\ &\qquad = \int_{S_\omega}\int_0^{+\infty} \varphi(\omega+s\omega')\left[\frac{L(\lambda p(\omega+s\omega'))}{l(\lambda)} - 1\right] s^{n-2}dsd\omega'\end{aligned}$$

Let $\Delta > 0$, we split the double integral as follows:

$$\frac{\hat{f}(\omega,\lambda p)}{\lambda^{\alpha+n-1}l(\lambda)} - p^{\alpha}\hat{\varphi}(\omega,1) \tag{3.2}$$

$$= \left\{\int_{S_\omega}\int_0^{\Delta} + \int_{S_\omega}\int_\Delta^{+\infty}\right\}\varphi(\omega+s\omega')\left[\frac{L(\lambda p(\omega+s\omega'))}{l(\lambda)} - 1\right] s^{n-2}dsd\omega' = I_1 + I_2.$$

We consider I_2 first; let $0<\beta<-\alpha-n+1$, then there are constants P and $M(\beta)$ such that

$$|L(\lambda p(\omega+s\omega'))/l(\lambda) - 1| \leq M(\beta)|p(\omega+s\omega')|^{\beta} = M(\beta)p^{\beta}(1+s^2)^{\beta/2}$$

for $p \geq P$ and $s \geq 0$. Hence,

$$|I_2| \leq p^{\beta}M(\beta)\int_{S_\omega}\int_\Delta^{+\infty}\varphi(\omega+s\omega')(1+s^2)^{\beta/2}s^{n-2}dsd\omega'. \tag{3.3}$$

As the order of φ is negative, we have the following estimate: for $\omega' \in S_\omega$, and $0 \leq s < +\infty$,

$$\varphi(\omega+s\omega') = s^{\alpha}\varphi(\omega/s+\omega') \leq s^{\alpha}\sup_{t\in S^{n-1}}\varphi(t).$$

Returning to (3.3) and absorbing the sup into $M(\beta)$ we have,

$$\begin{aligned}|I_2| &\le M(\beta)p^\beta \int_{S_\omega}\int_\Delta^{+\infty} s^{\alpha+n-2}(1+s^2)^{\beta/2}ds d\omega' \\ &\le 2M(\beta)p^\beta\omega_{n-2}\int_\Delta^{+\infty} s^{\alpha+\beta+n-2}ds = 2M(\beta)p^\beta\omega_{n-2}\frac{\Delta^{\alpha+\beta+n-1}}{\alpha+\beta+n-1}.\end{aligned}$$

For $\varepsilon > 0$, it follows that for sufficiently large fixed Δ, $|I_2| < \varepsilon/2$, uniformly in $\omega \in S^{n-1}$. Set $B^1(\Delta) = \{(\omega', s) \mid \omega' \in S_\omega,\ 0 \le s \le \Delta\}$, then for I_1, we have,

$$\begin{aligned}|I_1| &\le \sup_{B(\Delta)}\left\{\left|\frac{L(\lambda p(\omega+s\omega'))}{l(\lambda)} - 1\right|\right\}\int_{S_\omega}\int_0^\Delta \varphi(\omega+s\omega')s^{n-2}ds \\ &\le M(\Delta)\sup_{B(\Delta)}\left|\frac{L(\lambda p(\omega+s\omega'))}{l(\lambda)} - 1\right|,\end{aligned}$$

where $M(\Delta)$ depends on φ and Δ(fixed above). Since

$$\lim_{\lambda\to+\infty}\sup_{B(\Delta)}\left|\frac{L(\lambda p(\omega+s\omega'))}{l(\lambda)} - 1\right| = 0$$

uniformly in $\omega \in S^{n-1}$, it follows that for sufficiently large λ, we have $I_1 < \varepsilon/2$, uniformly in $\omega \in S^{n-1}$. This completes the proof.

As a corollary (of the proof) we have the following:

Corollary 3.1. *Let $f(x)$ be as in Theorem 3.1. Then,*

$$\hat{f}(\omega, p) \approx p^{\alpha+n-1}l(p)\hat{\varphi}(\omega, 1)$$

as $p \to +\infty$, uniformly in $\omega \in S^{n-1}$.

Developing an Abelian theorem for the Fourier transform requires expressing the above result for cones in $\mathbf{R}^n$. Let Γ be a cone in $\mathbf{R}^n$ i.e. for $x \in \Gamma$, $\lambda x \in \Gamma$ for every $\lambda > 0$. The dual cone is given by $\Gamma^* = \{y \in \mathbf{R}^n \mid x\cdot y \ge 0, \text{for all } x \in \Gamma\}$. We will assume Γ is convex and that the interior of Γ ($\int\Gamma$) and Γ^* are non-empty. For a cone Γ, we set $S_\Gamma^{n-1} = \Gamma \cap S^{n-1}$; $RV_\alpha(\Gamma)$ and $RV_\alpha(S_\Gamma^{n-1}\times\mathbf{R})$ denote the classes of regularly varying functions (at infinity) on Γ and $S_\Gamma^{n-1}\times\mathbf{R}$, respectively.

Theorem 3.2. *Let $\alpha \in \mathbf{R}$, let $\Gamma \subset \mathbf{R}^n$ be a cone and let $B \subset \int\Gamma^*$ be a closed cone. If $f \in RV_\alpha(\Gamma)$, then $\hat{f}(\omega,p)$ is defined for $\omega \in S_B^{n-1}$ $p > 0$ and $\hat{f} \in RV_\delta(S_B^{n-1}\times\mathbf{R})$ where $\delta = \alpha+n-1$.*

Proof. The proof is a modification of that of Theorem 3.1, we outline the modification. We must show

$$\lim_{\lambda\to+\infty}\sup_{\omega\in S_B^{n-1}}\left|\frac{\hat{f}(\omega,p)}{\lambda^{\alpha+n-1}l(\lambda)\hat{\varphi}(\omega,1)} - p^{\alpha+n-1}\right| = 0.$$

For $\omega \in S_B^{n-1}$, $\hat{f}(\omega,p)$ is easily seen to exist. We have,

$$\hat{f}(\omega,p) = \int_{S_\omega}\int_0^{r(p,\omega')} f(p\omega+s\omega)s^{n-2}dsd\omega',$$

where $r(p,\omega') = \sup\{s \mid p\omega + s\omega' \in \Gamma\} < +\infty$. The changes of variables used before gives

$$\hat{f}(\omega, p) = p^{n-1} \int_{S_\omega} \int_0^{r(p,\omega')/p} f(p(\omega + s\omega'))s^{n-2} ds d\omega'.$$

Simple geometry shows that $r(p,\omega')/p$ is uniformly bounded with respect to $\omega \in S_B^{n-1}$ above and below by positive constants. This geometrical fact allows one to carry through the proof like that of Theorem 3.1.

Remark. The quantity $r(p,\omega')/p$ is actually independent of p, this is useful later.

The Fourier sine transform of a function f living on a cone Γ is given by

$$\tilde{f}_S(\xi) = \int_\Gamma f(x) \sin(x \cdot \xi) dx.$$

Letting $\xi = s\omega$ and restricting attention to $\omega \in S_{\Gamma^*}^{n-1}$ gives:

$$\tilde{f}_S(s\omega) = \int_0^{+\infty} \int_{x\cdot\omega = p} f(x) dm(x) \sin spdp, \quad \text{or}$$

$$\tilde{f}_S(s\omega) = \int_0^{+\infty} \hat{f}(\omega, p) \sin spdp. \tag{3.5}$$

This is the analog of (1.4) for cones.

For convenience in theorem statement we iutroduce a subclass of regularly varying functions, $RVB_\alpha(\mathbf{R}^n)$. A regularly varying function R on Γ is said to be in the class $RVB_\alpha(\mathbf{R}^n)$ if the slowly varying function $L(x)$ associated with R has the property that for some $\beta > 0$, $|x|^\beta L(x)$ is bounded on every spherical wedge $\{x \in \Gamma \mid |x| \leq \delta\}$. We can now prove:

COROLLARY 3.2. *Let $-(n+1) < \alpha < -n$ and let B be a closed cone, $B \subset int\Gamma^*$. If $f \in RVB_\alpha(\Gamma)$ then $\tilde{f}_S$ is regularly varying at zero with index $\alpha + n - 2$ on B.*

Proof. By Theorem 3.2., $\hat{f} \in RV_\beta(S_B^{n-1} \times \mathbf{R})$ where $\beta = \alpha + n - 1$. Note that $-2 < \beta < -1$. We may write $\hat{f}(\omega, p) = p^{\alpha+n-1} l_1(\omega, p)\hat{\varphi}(\omega, 1)$ for some even slowly varying function l_1 on $S_B^{n-1} \times \mathbf{R}$. We need show that $p^\delta l_1(\omega, p)$ is bounded on every interval $[0, \Delta]$ uniformly in ω. Write:

$$\begin{aligned}\hat{f}(\omega, p) &= p^{\alpha+n-1} \int_{S_\omega} \int_0^{r(p,\omega')/p} \varphi(\omega + s\omega') L(p(\omega + s\omega'))s^{n-2} ds d\omega' \\ &= p^{-\delta+\alpha+n-1} \int_{S_\omega} \int_0^{r(p,\omega')/p} \varphi(\omega + s\omega')(1+s^2)^{-\delta/2} \\ &\quad \cdot [p^\delta (1+s^2)^{\delta/2} L(p(\omega + s\omega'))]^{n-2} ds d\omega'.\end{aligned}$$

Now, for $p \in [0, \Delta]$, and $0 \leq s \leq r(p,\omega')/p$ (notice there is no problem at $p = 0$, understanding $r(p,\omega')/p$ as a limit at that point), the point $p(\omega + s\omega')$ is contained in a spherical wedge $\{x \in \Gamma \mid |x| \leq \Delta\}$, for appropriate $\Delta > 0$. Hence the quantity in brackets above is bounded, say by M. Also,

$$\sup_{\omega' \in S_\omega} \varphi(\omega + s\omega'), \quad 0 \leq s \leq r(p,\omega')/p$$

is uniformly bounded for $\omega \in S_B^{n-1}$, say by M'. Then,

$$\hat{f}(\omega,p) \leq p^{-\delta+\alpha+n-1} M M' \int_{S_\omega} \int_0^{r(p,\omega')/p} s^{n-2}(1+s^2)^{-\delta/2} ds d\omega' \leq M p^{-\delta+\alpha+n-1},$$

where all constants are absorbed into M. From this it follows that $p^\delta l_1(\omega,p)$ is bounded on $[0,\Delta]$, uniformly in $\omega \in S_B^{n-1}$. By lemma 2.4, $\tilde{f}_S(s\omega)$ is regulary varying in s at zero with index $\alpha+n-2$, uniformly in $\omega \in S_B^{n-1}$. Lemma 2.5 concludes the proof.

4. The Dual Transform. In this section we develop a result dual to Theorem 3.1, the primary consequence is a Tauberian theorem. As before we must regulate the behavior of slowly varying functions $l(\omega,p)$ on $S^{n-1} \times \mathbf{R}$ in a neighborhood of $p=0$. A function $g \in RV_\alpha(S^{n-1} \times \mathbf{R})$ where $\alpha > -1$ is said to belong to the class $RVB_\alpha(S^{n-1} \times \mathbf{R})$ if it slowly varying part $l(\omega,p)$ satisfies: for some $0 < \delta < \alpha+1$, $p^\delta l(\omega,p)$ is bounded on every interval $p \in [0,\Delta]$, uniformly in $\omega \in S^{n-1}$ (i.e. $\varrho^\delta l(\omega,\varrho) \leq M(\Delta)$ for $p \in [0,\Delta]$, and all $\omega \in S^{n-1}$).

Theorem 4.1. *Let $\alpha > -1$. If $g \in RVB_\alpha(S^{n-1} \times \mathbf{R})$, then $\check{g}(x)$ is defined on $\mathbf{R}^n - \{0\}$ and $\check{g} \in RV_\alpha(\mathbf{R}^n)$.*

Proof. For brevity, we consider the case $g(\omega,\varrho) = l(p)\varphi(\omega,\varrho)$ i.e. the slowly varying part depends only on p. Let $S_+^{n-1} = \{\omega \in S^{n-1} \mid x\cdot\omega \geq 0\}$. As before, the existence of $\check{g}$ is left to the reader and,

$$\check{g}(x) = 2\int_{S_+^{n-1}} g(\omega, x\cdot\omega) d\omega,$$

and

$$\frac{\check{g}(\lambda x)}{\lambda^\alpha l(\lambda)} - \check{\varphi}(x) = 2\int_{S_+^{n-1}} (x\cdot\omega)^\alpha \left[\frac{l(\lambda(x\cdot\omega))}{l(\lambda)} - 1\right] \varphi(\omega,1) d\omega. \tag{4.1}$$

Let $B \subset \mathbf{R}^n - \{0\}$ be compact. We use the following basic fact about l [10]: for $\beta > 0$, there exists $M_1 = M_1(\beta)$ and $c = c(\beta)$ such that

$$|l(\lambda t)/l(\lambda) - 1| \leq M_1 \lambda^{-\beta} \tag{4.2}$$

for $c/\lambda \leq t \leq 1$ and $\lambda \geq c$. The integral in (4.1) is split as follows:

$$\begin{aligned}\frac{\check{g}(\lambda x)}{\lambda^\alpha l(\lambda)} - \check{\varphi}(x) &= 2\left\{\int_{0\leq x\cdot\omega\leq c/\lambda} + \int_{c/\lambda\leq x\cdot\omega\leq 1} + \int_{1\leq x\cdot\omega\leq |x|}\right\}(x\cdot\omega)^\alpha \\ &\quad\cdot \left[\frac{l(\lambda(x\cdot\omega))}{l(\lambda)} - 1\right]\varphi(\omega,1)d\omega. \\ &= I_1 + I_2 + I_3.\end{aligned}$$

For I_1 we have,

$$|I_1| \leq \frac{2}{l(\lambda)} \int_{0\leq x\cdot\omega\leq c/\lambda} (x\cdot\omega)^\alpha l(\lambda(x\cdot\omega))\varphi(\omega,1)d\omega + 2\int_{0\leq x\cdot\omega\leq c/\lambda} (x\cdot\omega)^\alpha \varphi(\omega,1)d\omega$$
$$= I_1' + I_1''.$$

Take M such that $|\varphi(\omega,1)| \leq M$, $\omega \in S^{n-1}$. Then,

$$I_1'' \leq 2M|x|^\alpha \int_0^{c/\lambda|x|} t^\alpha(1-t^2)^{(n-3)/2}dt \leq \frac{2Mc^{\alpha+1}}{\lambda^{\alpha+1}(\alpha+1)}|x|^{-1}.$$

Hence, for sufficiently large λ, $I_1'' \leq \varepsilon/4$, uniformly for $x \in B$. Now, taking $\beta = \delta$ in the hypothesis,

$$|I_1'| \leq M\frac{2\lambda^{-\beta}}{l(\lambda)} \left[\sup_{0\leq x\cdot\omega\leq c/\lambda} [\lambda(x\cdot\omega)]^\beta l(\lambda(x\cdot\omega))\right] \int_0^{c/\lambda|x|} t^{\alpha-\beta}dt$$
$$= \frac{2M}{\alpha-\beta+1}\frac{\beta^{\alpha-\beta+1}}{\lambda^{\alpha+1}l(\lambda)|x|^{\alpha-\beta+1}} \sup_{0\leq t\leq c} t^\beta l(t).$$

Since $\lambda^{\alpha+1}l(\lambda) \to +\infty$ as $\lambda \to +\infty$, again we have for sufficiently large λ, $I_1'' < \varepsilon/4$ uniformly for $x \in B$. For I_2, we apply (4.2):

$$|I_2| \leq 2MM_1\lambda^{-\beta}\int_{c/\lambda|x|}^{1/|x|} t^\alpha dt = \frac{2MM_1}{\alpha+1}\lambda^{-\beta}\left(1-\frac{c^{\alpha+1}}{\lambda^{\alpha+1}}\right)|x|^{-(\alpha+1)}.$$

Hence, for sufficiently large λ, $|I_2| < \varepsilon/4$, uniformly for $x \in B$. Finally, for I_3 we use the uniform convergence property of slowly varying functions:

$$|I_3| \leq \frac{2M}{\alpha+1}\sup_{1\leq x\cdot\omega\leq|x|}\left|\frac{l(\lambda(x\cdot\omega))}{l(\lambda)}-1\right|\cdot(1-|x|^{-\alpha-1})$$
$$= \frac{2M(1-|x|)^{-\alpha-1}}{\alpha+1}\sup_{1\leq t\leq|x|}\left|\frac{l(\lambda t)}{l(\lambda)}-1\right|.$$

Note $|I_3| < \varepsilon/4$ for sufficienty large λ, uniformly for $x \in B$. Combining estimates concludes the proof.

The above result gives the following Tauberian theorem for the Riesz potential. For $0 < \gamma < n$, the Riesz potential of order γ is defined by [3],

$$(I^\gamma f)(x) = C_n(\gamma)\int_{\mathbf{R}^n} |x-y|^{\gamma-n}f(x)dx,$$

where $C_n(\gamma)$ is a constant dependent on n and γ only.

Corollary 4.1. *Let $\alpha > -1$ and let $f : \mathbf{R}^n \to \mathbf{R}_+$ be measurable. If $\hat{f} \in RVB_\alpha(S^{n-1}\times\mathbf{R}$, then $(I^{n-1}f)(x)$ exists for $x \in \mathbf{R}^n - \{0\}$ and $I^{n-1}f \in RV_\alpha(\mathbf{R}^n)$.*

Proof. This follows from Theorem 4.1 and the formula [2]:

$$(I^{n-1}f)(x) = c(\hat{f})(x),$$

where c is a constant dependent only on n.

This result may also be written in the following way, the proof is like that of Corollary 3.2.

COROLLARY 4.2. *Let* $-n < \alpha < -n+1$. *If* $f \in RVB_\alpha(\mathbf{R}^n)$, *then* $(I^{n-1}f)(x)$ *exists on* $\mathbf{R}^n - \{0\}$ *and* $I^{n-1}f \in RV_\beta(\mathbf{R}^n)$ *where* $\beta = \alpha + n - 1$.

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(Received 16 04 1986)