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ON A FUNCTIONAL WHHICH IS QUADRATIC ON A-ORTHOGONAL VECTORS

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Abstract. Let X be a complex Hilbert space, $\dim X \geq 3$ and A be a bounded selfadjoint operator defined on X. We give a representation of a continuous functional H defined on X under the condition that H is quadratic on A-orthogonal vectors.

In [3] a continuous functional $F : X \to \Phi$ is studied which is additive on A-orthogonal vectors. Let us note that the square of functional which is additive on A-orthogonal vectors does not have to be quadratic. The purpose of this paper is to give a representation of the functional $H : X \to \Phi$ under the condition that is quadratic on A-orthogonal vectors. In [2] a representation is given in the case when $A = I$ (I denotes the identical operator).

The following theorem will be proved:

THEOREM 1. *Let H be a continuous functional defined on a (real or complex) Hilbert space X with $\dim X \geq 3$. Suppose that if $(x, Ay) = 0$ $(x, y \in X)$ then*

$$H(x+y) + H(x-y) = 2H(x) + 2H(y), \tag{$*$}$$

where $A : X \to X$ is a continuous selfadjoint operator with $\dim A(X) \neq 1, 2, 3$. Then there is a continuous linear operator B and quasi-linear continuous operator C and D such that

$$H(x) = (Bx, x) + (Cx, x) + (x, Dx) \tag{$**$}$$

for all $x \in X$.

We will use the same technique as in [2] and the proof of the theorem will be based upon the following lemmas.

LEMMA 1. *Under the hypotheses of Theorem 1 there exist functionals $B(x)$, $C(x)$ and $D(x)$ (defined on X) satisfying $(*)$ such that for all complex numbers λ and for all x in X:*

$$B(\lambda x) = |\lambda|^2 B(x), \quad C(\lambda x) = \lambda^2 C(x), \quad D(\lambda x) = \bar{\lambda}^2 D(x)$$

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Moreover, $H(x) = B(x) + C(x) + D(x)$.

Proof. We first show that for the functional $H(x)$ we have $H(rx) = r^2 H(x)$ for all $x \in X$, where r is a real number. It is obvious that $H(0) = 0$.

1° Let $(x, Ax) = 0$ for some $x \in X$ $(x \neq 0)$; then, applying relatio $(*)$, we obtain $H(2x) = 2^2 H(x)$. Thus

$$\begin{aligned} H(3x) + H(x) &= 2H(2x) + 2H(x), \qquad H(3x) = 2H(2x) + H(x) \\ &= 2 \cdot 2^2 H(x) + H(x), \quad H(3x) = 3^2 H(x). \end{aligned}$$

Similarly we obtain $H(4x) = 4^2 H(x)$, $H(5x) = 5^2 H(x), \ldots$. Suppose that $H(nx) = n^2 H(x)$ holds for a natural number n. We shall prove that $H[(n+1)x] = (n+1)^2 H(x)$. For this we have:

$$\begin{aligned} H[(n+1)x] + H[(n-1)x] &= 2H(nx) + 2H(x) \\ H[(n+1)x] &= 2H(nx) + 2H(x) - H(n-1)x \\ &= 2n^2 H(x) + 2H(x) - (n-1)^2 H(x). \end{aligned}$$

$$H[(n+1)x] = [2n^2 + 2 - (n-1)^2] H(x), \quad H[(n+1)x] = (n+1)^2 H(x).$$

Thus, $H(nx) = n^2 H(x)$ holds for all natural n.

Similarly we obtain $H(nx) = n^2 H(x)$, if $n = -1, -2, -3, \ldots$. It also follows easily (because of the continuity of H) that $H(rx) = r^2 H(x)$ for all real r.

2° Let $(Ax, x) \neq 0$. Then there exist a $y \in X$ $(y \neq 0)$ such that $(x, Ay) = 0$ and $(Ay, y) = \pm (Ax, x)$.

(a) If $(Ay, y) = (Ax, x)$, then the vectors $nx + y$ and $x - ny$ are pairwise A-orthgonal. According to $(*)$ we can write

$$H[(nx+y) + (x-ny)] + H[(nx+y) - (x-ny)] = 2H(nx+y) + 2H(x-ny), \quad (1)$$

$$H[(n+1)x - (n-1)y] + H[(n-1)x + (n+1)y] = 2H(nx+y) + 2H(x-ny), \text{ or}$$

$$H[(n+1)y - (n-1)x] + H[(n-1)y + (n+1)x] = 2H(ny+x) + 2H(y-nx) \quad (2)$$

If we add (1) and (2) and take into consideration $(*)$, we get

$$\begin{aligned} &2H[(n+1)x] + 2H[(n-1)y] + 2H[(n-1)x] + 2H[(n+1)y] \\ &\quad = 4H(nx) + 4H(y) + 4H(x) + 4H(ny) \end{aligned}$$

or

$$\begin{aligned} &H[(n+1)x] + H[(n-1)y] + H[(n-1)x] + H[(n+1)y] \\ &\quad = 2H(nx) + 2H(y) + 2H(x) + 2H(ny). \end{aligned}$$

Let

$$H(kx) + H(ky) = k^2 [H(x) + H(y)] \qquad (3)$$

hold for all $k = 0, 2, 3, \dots, n$. It is easy to prove that (3) is true for $n = k + 1$. In [1] it has been proved that there exists a $z \in X$ such that $(x, Az) = (y, Az) = 0$ and $(Ax, x) = (Ay, y) = (Az, z)$, and on the basis of (3) we can write

$$H(nx) + H(ny) = n^2[H(x) + H(y)] \tag{3'}$$

$$H(nx) + H(nz) = n^2[H(x) + H(z)] \tag{3''}$$

$$H(ny) + H(nz) = n^2[H(y) + H(z)]. \tag{3'''}$$

Subtracting (3‴) from (3″), we obtain $H(nx) - H(ny) = n^2[H(x) - H(y)]$, which together with (3′) gives $H(nx) = n^2 H(x)$. Due to the continuity of the functional H, $H(rx) = r^2 H(x)$ holds for all real numbers r.

(b) Let $(Ay, y) = -(Ax, x)$. It follows that $(A(x \pm y, x \pm y) = 0$ and according to 1° we get $H[n(x + y)] = n^2 H(x + y)$, $H[n(x - y)] = n^2 H(x - y)$. Besides that we have $H[n(x + y)] + Hn(x - y)] = 2H(nx) + 2H(ny)$ or

$$n^2 H(x + y) + n^2 H(x - y) = 2H(nx) + 2H(ny). \tag{4}$$

In [1] it has been shown that there exists a $z \in X$ such that $(Ax, z) = 0$ and $(Az, z) = -(Ax, x), (Ay, z) = 0$, $(Ay, y) = (Az, z)$. On the basis of (a) we can write

$$n^2 H(y + z) + n^2 H(y - z) = 2H(ny) + 2H(nz) = 2n^2 H(y) + 2n^2 H(z) \tag{5}$$

or

$$n^2 H(x + z) + n^2 H(x - z) = 2H(nx) + 2H(nz). \tag{6}$$

If we subtract (5) from (6), we get

$$n^2 H(x + z) + n^2 H(x - z) - n^2 H(y + z) - n^2 H(y - z) = 2H(nx) - 2H(ny).$$

If we add this last relation to (4) we obtain

$$\begin{aligned} n^2 H(x + z) + n^2 H(x - z) - n^2 H(y + z) - n^2 H(y - z) + n^2 H(x + y) \\ + n^2 H(x - y) = 4H(nx) \end{aligned}$$

or

$$2n^2 H(x) + 2n^2 H(z) - 2n^2 H(y) - 2n^2 H(z) + 2n^2 H(x) + 2n^2 H(y) = 4H(nx)$$

or $H(nx) = n^2 H(x)$. Since the functional H is continuous then $H(rx) = r^2 H(x)$ holds for all real numbers r. Therefore $H(rx) = r^2 H(x)$ holds for all real numbers r nad for each $x \in X$.

Let $2B(x) = H(ix) + H(x)$. It is easy to see that $B(x)$ is a continuous and quadratic functional on A-orthogonal vectors, as well as it satisfies $B(rx) = r^2 B(x)$, that $B(ix) = B(x)$.

1° Let $(Ax, x) = 0$ for some $x \in X$. Then $(A\alpha x, i\beta x) = 0$ (α, β real numbers). For $\lambda = \alpha + i\beta$ we have

$$\begin{aligned} B(\lambda x) + B(\bar{\lambda} x) &= B((\alpha + i\beta)x) + B((\alpha - i\beta)x) = B(\alpha x + i\beta x) + B(\alpha x - i\beta x) \\ &= 2B(\alpha x) + 2B(i\beta x) = 2\alpha^2 B(x) + 2\beta^2 B(ix) \\ &= 2\alpha^2 B(x) + 2\beta^2 B(x) = 2(\alpha^2 + \beta^2) B(x) = 2|\lambda|^2 B(x). \end{aligned}$$

Hence $B(\lambda x) + B(\bar{\lambda} x) = 2|\lambda|^2 B(x)$.

2° Let $(Ax, x) \neq 0$. Then there exists a $y \in X$ such that $(x, Ay) = 0$ and $(Ay, y) = \pm(Ax, x)$. Let us consider the case when (a) $(Ay, y) = (Ax, x)$. Then if $\lambda = \alpha + i\beta$ (α, β real) and $e_1 = (x+y)/2$, $e_2 = (x-y) \mid 2i$, it follows that

$$B(\lambda x) + B(\lambda y) + B(\bar{\lambda} x) + B(\bar{\lambda} y) = 2|\lambda|^2 [B(x) + B(y)].$$

We can select a $z \in X$ such that $(x, Az) = 0$, $(y, Az) = 0$ and $(x, Ax) = (y, Ay) = \pm(z, Az)$. Let us consider the case when the sign is $\pm$. By analogy with the equation above, we can write the following.

$$B(\lambda x) + B(\lambda z) + B(\bar{\lambda} x) + B(\bar{\lambda} z) = 2|\lambda|^2 [B(x) + B(z)]$$
$$B(\lambda y) + B(\lambda z) + B(\bar{\lambda} y) + B(\bar{\lambda} z) = 2|\lambda|^2 [B(y) + B(z)].$$

From the last three equalities we have $B(\lambda x) + B(\bar{\lambda} x) = 2|\lambda|^2 B(x)$.

Let us consider the case when (b) $(Ay, y) = -(Ax, x)$. Then $(A(x \pm y), x \pm y) = 0$. On the basis of 1° we have

$$B(\lambda(x+y)) + B(\bar{\lambda}(x+y)) = 2|\lambda|^2 B(x+y)$$
$$B(\lambda(x-y)) + B(\bar{\lambda}(x-y)) = 2|\lambda|^2 B(x-y).$$

Summing these two equations we obtain

$$B(\lambda(x+y)) + B(\lambda(x-y)) + B(\bar{\lambda}(x+y)) + B(\bar{\lambda}(x-y)) = 2|\lambda|^2 (B(x+y) + B(x-y))$$

or

$$2B(\lambda x) + 2B(\lambda y) + 2B(\bar{\lambda} x) + 2B(\bar{\lambda} y) = 4|\lambda|^2 B(x) + 4|\lambda|^2 B(y)$$

or

$$B(\lambda x) + B(\lambda y) + B(\bar{\lambda} x) + B(\bar{\lambda} y) = 2|\lambda|^2 (B(x) + B(y)). \tag{7}$$

As before, there exists a $z \in X$ such that $(Ax, z) = (Ay, z) = 0$, $(Ay, y) = (Az, z)$ and $(Az, z) = -(Ax, x)$. We have

$$B(\lambda x) + B(\lambda z) + B(\bar{\lambda} x) + B(\bar{\lambda} z) = 2|\lambda|^2 (B(x) + B(y)) \tag{8}$$
$$B(\lambda y) + B(\lambda z) + B(\bar{\lambda} y) + B(\bar{\lambda} z) = 2|\lambda|^2 (B(y) + B(z)) \tag{9}$$
$$B(\lambda x) - B(\lambda y) + B(\bar{\lambda} x) - B(\bar{\lambda} y) = 2|\lambda|^2 B(x) - 2|\lambda|^2 B(y). \tag{10}$$

From (7) and (10) it follows that $B(\lambda x) + B(\bar{\lambda} x) = 2|\lambda|^2 B(x)$. Thus from these considerations we can conclude that for each $x \in X$ and each complex λ we have

$$B(\lambda x) + B(\bar{\lambda} x) = 2|\lambda|^2 B(x) \tag{11}$$

If in (11) we replace λ by $e^{i\varphi}$ (φ real) and ix by $e^{i\varphi}x$, we obtain $B(e^{2i\varphi}x) + B(x) = 2B(e^{i\varphi}x)$. Similarly we get $B(e^{-2i\varphi}x) + B(x) = 2B(e^{-i\varphi}x)$. Thus we have

$$B(e^{2i\varphi}x) - B(e^{-2i\varphi}x) = 2[B(e^{i\varphi}x) - B(e^{-i\varphi}x)]. \tag{12}$$

For fixed $x \in X$ let us set

$$I(\alpha) = B(\alpha x) - B(\alpha^{-1} x) \quad (\alpha = e^{i\varphi}). \tag{13}$$

It is easy to show that $I(\alpha) = 0$ for all $\alpha = e^{i\varphi}$ (φ real). From this fact it follows that $B(\bar{\lambda} x) = B(\lambda x)$, and from that (due to (11)) we have

$$B(\lambda x) = |\lambda|^2 B(x) \quad (x \in X \text{ and } \lambda\text{-complex}).$$

Let us put

$$2S(x) = H(ix) - H(x). \tag{15}$$

The functional $S(x)$ is continuous, quadratic on A-orthogonal vectors and quadratic homogenous, i.e. $S(rx) = r^2 S(x)$, and besides that

$$S(ix) = -S(x), \quad (x \in X). \tag{16}$$

In the same way as with the functinal $B(x)$, we obtain

$$S(\lambda x) + S(\bar{\lambda} x) = (\lambda^2 + \bar{\lambda}^2) S(x) \tag{17}$$

for each x in X and for each λ. If in (17) we put $\lambda = \alpha$ ($|\alpha| = 1$, $\alpha^{4n} \neq 1$, $n = 1, 2, \dots$) and αx instead of x, we obtain

$$S(\alpha^2 x) + S(x) = (\alpha^2 + \bar{\alpha}^2) S(x) \tag{17'}$$

or

$$\alpha^4/(\alpha^8 - 1) \cdot [S(\alpha^2 x) - \bar{\alpha}^4 S(x)] = \alpha^2/(\alpha^4 - 1) \cdot [S(\alpha x) - \alpha^2 S(x)].$$

By induction we can prove

$$\alpha^{2n}/(\alpha^{4n} - 1) \cdot [S(\alpha^n x) - \alpha^{2n} S(x)] = \alpha^2/(\alpha^4 - 1) \cdot [S(\alpha x) - \bar{\alpha}^2 S(x)].$$

If $\beta = \alpha^n$, then

$$\beta^2/(\beta^4 - 1) \cdot [S(\beta x) - \bar{\beta}^2 S(x)] = \alpha^2/(\alpha^4 - 1) \cdot [S(\alpha x) - \bar{\alpha}^2 S(x)]$$

or

$$1/(\beta^2 - \bar{\beta}^2) \cdot [S(\beta x) - \bar{\beta}^2 S(x)] = 1/(\alpha^2 - \bar{\alpha}^2) \cdot [S(\alpha x) - \bar{\alpha}^2 S(x)]. \tag{17''}$$

In the last relation α and β are arbitrary numbers such that $|\alpha| = |\beta| = 1$, $\alpha^4 \neq 1$ and $\beta^4 \neq 1$ and (17) holds for each x in X. Since $S(rx) = r^2 S(x)$, from (17$''$) it follows immediately that

$$[S(\lambda x) - \bar{\lambda}^2(x)]/(\bar{\lambda}^2 - \lambda^2) = [S(\lambda_1 x) - \bar{\lambda}_1^2 S(x)]/(\lambda_1^2 - \bar{\lambda}_1^2) \tag{17'''}$$

($\lambda^2 \neq \bar{\lambda}^2$, $\lambda_1^2 = \bar{\lambda}_1^2$) for all x in X.

The right-hand side of relation (17$'''$) is constant, for any λ ($\lambda^2 = \bar{\lambda}^2$) and if for fixed λ_1 we put

$$C(x) = [\bar{\lambda}_1^2 S(x) - S(\lambda_1 x)]/(\lambda_1^2 - \bar{\lambda}_1^2)$$

we obtain $C(\lambda x) = [\bar{\lambda}_1^2 S(\lambda x) - S(\lambda_1 \lambda x)]/(\lambda_1^2 - \bar{\lambda}_1^2)$. According to (17‴) we conclude that $C(x)$ and $C(\lambda x)$ do not depend on λ_1, and if we put $\lambda_1 = \lambda$ (in the relation for $C(x)$), $\lambda_1 = \bar{\lambda}$ (in the relation for $C(\lambda x)$), we obtain $C(\lambda x) = \lambda^2 C(x)$, for each complex λ and x in S. Let us put $D(x) = -S(x) - C(x) = [S(\lambda_1 x) - \lambda_1^2 S(x)]/(\lambda_1^2 - \bar{\lambda}_1^2)$. Then it follows that $D(\lambda x) = \bar{\lambda}^2 D(x)$ (x in X, λ a complex number). Since $H(x) = B(x) - S(x)$ and $-S(x) = C(x) + D(x)$ it follows that $H(x) = B(x) + C(x) + D(x)$ Q.E.D.

Lemma 2. *Suppose that the functional H satisfies the conditions of THeorem 1 and that*

$$H(\lambda x) = |\lambda|^2 H(x) \tag{18}$$

for all in X and for every complex number λ. Then there exists a unique continuous linear operator B such that for all x in X

$$H(x) = (Bx, x). \tag{19}$$

Proof. Let us put

$$F(x,y) = H(x+y) - H(x-y) \quad (x, y \text{ in } X) \tag{20}$$

Let further

$$X_y = \{x \mid x \in X,\ (Ax, y) = 0\}. \tag{21}$$

For a fixed y and for x in X, $F(x,y)$ is a continuous functional (on X) and moreover from $(x, Az) = 0$, x, z in X it follows that $F(x+z, y) = F(x,y) + F(z,y)$. On the basis of [3] there exist unique vectors a_y and b_y in X_y and a unique complex number α_y such that

$$F(x,y) = 2(a_y, x) + 2(x, b_y) + 2\alpha_y (Ax, x) \tag{22}$$

for all x in X_y. Since the functional H is quadratic on A-orthogonal vectors we have

$$H(x+y) = H(x) + H(y) + (a_y, x) + (x, b_y) + \alpha_y (Ax, x), \quad ((Ax, y) = 0). \tag{23}$$

1° Let $x \in X$ be such that $(Ax, x) = 0$. Then the relation (23) has a form

$$H(x+y) = H(x) + H(y) + (a_y, x) + (x, b_y), \quad ((Ax, y) = 0).$$

2° Let $x \in X$ be such that $(Ax, x) \neq 0$. Then due to the continuity of the functional F we conclude that $\alpha_y = 0$, and relation (23) becomes

$$H(x+y) = H(x) + H(y) + (a_y, x) + (x, b_y), \quad ((Ax, y) = 0). \tag{23'}$$

We can write the space X as the direct sum of orthogonal and A-orthogonal invariant subspaces X^0, X^-, X^+ of the operator A, where $X^0 = \{(\in X \mid Ax = 0\}$. In X^- it holds that $(Ax, x) < 0$ for $x \neq 0$, and in X^+ it holds that $(Ax, x) > 0$ for $x \neq 0$. In each of these subspaces we can select a maximal A-orthonormal system. Let $\{e_i\}$ be a maximal A-orthonormal system in the space X, which is equal to the union of these maximal A-orthonormal systems. Let us take an arbitrary x in X; then $x = \sum_{i=1}^{\infty} \alpha_i e_i$. Let us put $x_n = \sum_{i=1}^{n} \alpha_i e_i$. Applying relation (23′) we obtain $H(x_n) = H_1(x_n) + H_2(x_n)$ where $H_1(x_n) = \sum_{i=1}^{n} |\alpha_i|^2 H(e_i)$; $H_2(x_n) = \sum_{k=1}^{n-1} [(\alpha_k a_k,\, \bar{x}_{k+1}) + (\bar{x}_{k+1}, \alpha_k b_k)]$, $a_k = a_{e_k}$, $b_k = b_{e_k}$, $\bar{x}_k = \sum_{i=k}^{n} \alpha_i e_i$ $(1, 2, \dots, n-1)$. We claim that $H_1(x_n)$ and $H_2(x_n)$ are quadratic on vectors of the form $x_n = \sum_{i=1}^{n} \alpha_i e_i$. Let $x_n = \sum_{i=1}^{n} \alpha_i e_i$, $y_m = \sum_{i=1}^{m} \beta_i e_i$ (Set $n = \max\{n, m\}$). Then[1]

$$
\begin{aligned}
&H_1(x_n + y_m) + H_1(x_n - y_m) = H_1(\textstyle\sum(\alpha_i + \beta_i)e_i) + H_1(\textstyle\sum(\alpha_i - \beta_i)e_i)\\
&= \textstyle\sum(|\alpha_i + \beta_i|^2 H(e_i) + \sum|\alpha_i - \beta_i|^2 H(e_i) = \sum[|\alpha_i + \beta_i|^2 + |\alpha_i - \beta_i|^2] H(e_i)\\
&= \textstyle\sum(2|\alpha_i|^2 + 2|\beta_i|^2) H(e_i) = 2\sum|\alpha_i|^2 H(e_i) + 2\sum|\beta_i|^2 H(e_i) = 2H_1(x_n) + 2H_1(y_m).
\end{aligned}
$$

Thus, $H_1(x_n + y_m) + H_1(x_n - y_m) = 2H_1(x_n) + 2H_1(y_m)$. Similarly it can be proved that

$$H_2(x_n + y_m) + H_2(x_n - y_m) = 2H_2(x_n) + 2H_2(y_m).$$

Therefore for all vectors $x_n = \sum \alpha_i e_i$, $y_m = \sum \beta_i e_i$,

$$H(x_n + y_m) + H(x_n - y_m) = 2H(x_n) + 2H(y_m).$$

Thus the functional H is quadratic on the set $S = \{x_n \mid x_n = \sum \alpha_i e_i,\ e_i - A\text{-orthonormal vectors}\}$. Taking into consideration that the set S is everywhere X-dense and that H is a continuous functional, the equation $H(x+y) + H(x-y) = 2H(x) + 2H(y)$ holds for x, y in X. Hence Lemma 2 follows from (18) and the continuity of H. Q.E.D.

LEMMA 3. *If the functional H satisfies the conditions of Theorem 1 and moreover*

$$H(\lambda x) = \lambda^2 H(x) \quad (\text{or } H(\lambda x) = \bar{\lambda}^2 H(x))$$

holds for every complex number λ and all $x \in X$, then

$$H(x+y) + H(x-y) = 2H(x) + 2H(y) \text{ holds for all } x, y \in X.$$

Proof. 1° Let $(Ax, y) = 0$ for some x, y in X. Then due to the hypothesis the statement holds.

2° Let $(Ax, y) \neq 0$ for some x, y in X[2]. We can suppose that $(Ax, x) \neq 0$. Then there exists a $z \in X$ such that $(Az, z) \neq 0$ and $(Ax, z) = 0$, $(Ay, z) = 0$. We can write $H(x+z) + H(x-z) = 2H(x) + 2H(z)$, $H(x+iz) + H(x-iz) = 2H(x) - 2H(z)$.

Thus

$$4H(x) = H(x+z) + H(x-z) + H(x+iz) + H(x-iz). \tag{24}$$

[1] $\sum$ means $\sum_{i=1}^{n}$.

[2] We can suppose that $(Ax, x) \neq 0$

Let us select the number α such that $(Ax+y), x-y) + \bar{\alpha}(Az, z) = 0$. Taking this condition into consideration we obtain

$$(A(x+y+z), x-y+\alpha z) = 0, \quad (A(x+y-z), x-y-\alpha z) = 0$$
$$(A(x+y+iz), x-y+\alpha iz) = 0, \quad (A(x+y-iz), x-y-\alpha iz) = 0.$$

Applying relation (24) we get

$$4H(x+y) = H(x+y+z) + H(x+y-z) + H(x+y+iz) + H(x+y-iz)$$
$$4H(x-y) = H(x-y+z) + H(x-y-z) + H(x-y+iz) + H(x-y-iz).$$

Now making use of the fact that the functional H is quadratic on A-orthogonal vectors we obtain

$$H(x+y) + H(x-y) = 2H(x) + 2H(y).$$

This holds when $(A(x+y), x-y) \neq 0$. If $(A(x+y), x-y) = 0$, the statement obviously holds. From 1° and 2° we conclude that $H(x+y)+H(x-y) = 2H(x)+2H(y)$ holds for $x, y \in X$. Now, let us consider the functional H with the property $H(\lambda x) = \lambda^2 H(x)$. As with the proof of Lemma 2 it is also easy to show that

$$F(x,y) = 2(a_y, x) + 2(x, b_y) \tag{25}$$

and that $F(x,y) = H(x+y) - H(x-y)$. Relation (25) holds for all $x, y \in X$. Besides that

$$F(x,y) = H(x+y) - H(x-y) = H(y+x) - H(y-x) = F(y,x)$$

and

$$\begin{aligned} F(x_1+x_2, y) &= 2(a_y, x_1+x_2) + 2(x_1+x_2, b_v) \\ &= 2(a_y, x_1) + 2(x_1, b_y) + 2(a_y, x_2) + 2(x_2, b_y) \\ &= F(x_1, y) + F(x_2, y). \end{aligned}$$

Thus

$$F(x_1+x_2, y) = F(x_1, y) + F(x_2, y), \quad F(x, y_1+y_2) = F(x, y_1) + F(x, y_2).$$

It is now easy to obtain

$$a_{y_1+y_2} = a_{y_1} + a_{y_2} \quad b_{y_1+y_2} = b_{y_1} + b_{y_2} \tag{26}$$
$$a_{\lambda y} = \bar{\lambda}^2/\lambda \cdot a_y \qquad (\lambda \text{ – complex number} \neq 0)$$
$$b_{\lambda y} = \bar{\lambda} b \tag{27}$$

since $a_y = 0$ for y in X. Thus $H(x+y) - H(x-y) = (x, b_y)$ holds for all $x, y \in X$. For $x = y$, $H(2x) = (x, b_x)$ or $H(x) = (x, Dx)$ where $Dx = b_x/4$ and D is a quasi-linear operator. For $H(x)$ instead of $H(\lambda x) = \lambda^2 H(x)$, the condition $H(\lambda x) = \bar{\lambda}^2 H(x)$, should be added and it is easy to show in this way that $H(x) = (Cx, x)$, and that C is a quasi-linear operator. Continuity of D and C is clear. So, we obtain

LEMMA 4. *If the functional H satisfies the conditions of Lemma 2, there exists a unique continuous quasi-linear operator $D(C)$ such that*

$$H(x) = (x, Dx), \quad (H(x) = (Cx, x)).$$

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