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A GENERALIZATION OF EQUIVALENCE RELATIONS

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Abstract. A many sorted generalization, called *case relation*, of the notion of equ'valence relation is given. Some fundamental properties of the case relations are proved, such as: some set theoretical characterizations and a formula which describes all case relations of the given sets.

By the set-theoretical interpretation of natural languages with highly developed inflection introduced in [2], the verbs have been interpreted as many-sorted relations of the given domains. Namely, if $\mathcal{D}$ is an interpretation, α a verb having the mark $\langle k_1, k_2, \ldots, k_n\rangle$, $i \in D_s$, then the meaning $m(i)(\alpha)$ of the verb α at the index i has been defined as an n-ary relation of the sets $D_{k_1}, D_{k_2}, \ldots, D_{k_n}$ i.e. as a subset of $D_{k_1} \times D_{k_2} \times \cdots \times D_{k_n}$. In what follows for these relations we use the name *case relations*. Case relations have various properties which resemble the well-known properties of the corresponding one-sorted relations. For example, consider the predicate

(1) *is similar*

(in Serbocroatian: *je sličan*, in German: *ist änlich*).

In most inflective languages (1) has the mark $\langle 1, 3\rangle$, which means that it is applicable to an ordered pair of nouns wich are in nominative and dative respectively. Thus, in the interpretation $\mathcal{D}$ the corresponding case relation for each chosen $i \in D_s$ is a subset of $D_1 \times D_3$. Denote this relation by $\sim$. On the basis of the usual properies of (1) it follows immediately that $\sim$ satisfies:

(R_{13}) $\quad x^1 \sim x^3$

(S_{13}) $\quad x^1 \sim y^3 \Rightarrow y^1 \sim x^3$

(T_{13}) $\quad x^1 \sim y^3 \wedge y^1 \sim z^3 \Rightarrow x^1 \sim z^3 \qquad (x^1, y^1, z^1 \in D_1, x^3, y^3, z^3 \in D_3)$.

Obviously, (R_{13}), (S_{13}), (T_{13}) are generalizations of reflexivity, symmetry and transitivity. For that reason we call the relation $\sim$ having the properties (R_{13}), (S_{13}),

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(T_{13}), or more generally the properties:

(R_{ij}) $\quad x^i \sim x^j$

(S_{ij}) $\quad x^i \sim y^j \Rightarrow y^i \sim x^j$

(T_{ij}) $\quad x^i \sim y^j \wedge y^i \sim z^j \Rightarrow x^i \sim z^j \qquad (x^i, y^i, z^i \in D_i, x^j, y^j, z^j \in D_j).$

a case equivalence relation or more precisely an (i,j)-*case equivalence relation*. Similarly, the notion of an order relation can be generalized to the notion of *a case order relation*.

In the sequel we develop a small theory of (f,g)-*equivalence relations* which comprehends the notion of case equivalence relations.

Definition 1. Let A, B be nonempty sets, $f : A \to B$ a one-to-many[1] mapping from A to B, $g : B \to A$ a one-to-many mapping from B to A and $\sim \subseteq A \times B$ a binary case relation. We say that $\sim$ is an (f,g)-equivalence relation iff:

(i) Both f, g are *onto*:

(0_f) $\qquad (\forall y \in B)(\exists x \in A) y = f(x)$

(0_g) $\qquad (\forall x \in A)(\exists y \in B) x = g(y);$

(ii) $\sim$ *does not discern* f and g, i.e. for each $f(y)_i$, $g(x)_i$

$$(D_{f,g}) \qquad \begin{aligned} &x \sim f(y)_1 \Leftrightarrow x \sim f(y)_2, &&(x, y \in A)\\ &g(x_1) \sim y \Leftrightarrow g(x)_2 \sim y &&(x, y \in B); \end{aligned}$$

(iii) f, g are $\sim$-*inverse* to each other:

$(I_{f,g})$ $\qquad x \sim y \Leftrightarrow g(f(x)) \sim y, \quad x \sim y \Leftrightarrow x \sim f(g(y)) \qquad (x \in A, y \in B);$

(iv) $\sim$ is (f,g)-*reflexive* (f,g)-*symmetric* and (f,g)-*transitive*:

$(R_{f,g})$ $\qquad x \sim f(x),\ g(y) \sim y$

$(S_{f,g})$ $\qquad x \sim y \Leftrightarrow g(y) \sim f(x)$

$(T_{f,g})$ $\qquad x \sim y \wedge g(y) \sim z \Leftrightarrow x \sim z, \quad$ for all $x \in A,\ y \in B.$

For example, let $A = \{a_1, b_1, c_1, d_1, e_1\}$, $B = \{a_2, b_2, c_2\}$, $f = \{(a_1, a_2), (b_1, b_2), (c_1, a_2), (d_1, c_2), (e_1, c_2)\}$; $g = \{(a_2, a_1), (a_2, b_1), (b_2, b_1), (b_2, c_1), (c_2, d_1), (c_2, e_1)\}$ and let $\sim = \{(a_1, a_2), (a_1, b_2), (b_1, a_2), (b_1, b_2), (c_1, a_2), (c_1, b_2), (d_1, c_2), (e_1, c_2)\}$. It is not difficult to verify that $\sim$ is an (f,g)-equivalence relation.

In the case $A = B$ and f, g are identity mappings of A, Definition 1 reduces to the definition of an equivalence relation of A. If f is a one-to-one mapping wich is 1 - 1 and *onto*, and g is f^{-1}, the conditions for an (f, f^{-1})-equivalence relation read:

$$\begin{aligned} &x \sim f(x),\ f^{-1}(y) \sim y,\\ &x \sim y \Rightarrow f^{-1}(y) \sim f(x),\\ &x \sim y \wedge f^{-1}(y) \sim z \Rightarrow x \sim z. \end{aligned}$$

[1] i.e. f is a subset of $A \times B$ having the property: $(\forall x \in A)(\exists y \in B)(x, y) \in f$. By $f(x), f(x)_0, f(x)_1, \ldots$ we denote different images of $x \in A$ by the mapping f.

Especially, if $A = D_i$, $B = D_j$ and f maps x^i to x^j the definition of an (f, f^{-1})-equivalence relation reduces to the definition of the (i,j)-case equivalence relation.

The following theorem is an immediate consequence of Definition 1.

THEOREM 1. *The (f,g)-equivalence relation $\sim$ has the following properties:*

$$x \sim y \wedge x \sim z \Rightarrow g(y) \sim z. \tag{3}$$

$$x \sim y \Leftrightarrow (\forall z \in B)(x \sim z \Leftrightarrow g(y) \sim z), \tag{4}$$

$$x \sim y \wedge z \sim y \Rightarrow z \sim f(x), \tag{5}$$

$$x \sim y \wedge z \sim f(x) \Rightarrow z \sim y, \tag{6}$$

$$x \sim y \Leftrightarrow (\forall z \in A)(z \sim y \Leftrightarrow z \sim f(x)). \tag{7}$$

Starting from the (f,g)-equivalence relation $\sim$, we define two binary relations $\sim_A$, $\sim_B$, of the sets A, B respectively.

Definition 2. $x \sim_A y \Leftrightarrow x \sim f(y), \quad (x, y \in A)$
$x \sim_B y \Leftrightarrow g(x) \sim y, \quad (x, y \in B).$

In virtue of $(D_{f,g})$, it follows that $\sim_A$, $\sim_B$ do not depend on the choice of $f(y)$, $g(x)$, and therefore the definitions are correct. The properties of $\sim_A$, $\sim_B$ are summarized in the following theorem.

THEOREM 2. (i) *$\sim_A$, $\sim_B$ are equivalence relations of the sets A, B respectively.*

(ii) *Neiher $\sim_A$ discerns g nor $\sim_B$ discerns f, i.e.*

$$f(x)_1 \sim_B f(x)_2,\ g(y)_1 \sim_A g(y)_2,\ \textit{for all}\ f(x)_i,\ g(y)_i,\ x \in A,\ y \in B.$$

(iii)
$$x \sim_A g(f(x)),\ y \sim_B f(g(y))\ \textit{for all}\ \ x \in A, y \in B.$$

(iv) *$\sim_A$, $\sim_B$ are compatible with f, g respectively, i.e.*

$$x \sim_A y \Rightarrow f(x) \sim_B f(y), \quad (x, y \in A)$$
$$x \sim_B y \Rightarrow g(x) \sim_A g(y), \quad (x, y \in B).$$

Proof. Part (i) follows by $(R_{f,g})$, $(S_{f,g})$, $(I_{f,g})$, and by part (5) of Theorem 1. (ii) follows by $(D_{f,g})$ and (iii) by $(I_{f,g})$. The proof of the first implication in (iv) reads

$$x \sim_A y \Rightarrow x \sim f(y) \Rightarrow g(f(y)) \sim f(x) \Rightarrow f(x) \sim_B f(y).$$

We can prove similarly the second implication.

Using $(I_{f,g})$ it follows immediately that the implications in part (iv) may be replaced by equivalences.

THEOREM 3.

$$\begin{aligned} x \sim_A y &\Leftrightarrow f(x) \sim_B f(y) \quad (x, y \in A) \\ x \sim_B y &\Leftrightarrow g(x) \sim_A g(y) \quad (x, y \in B) \end{aligned} \tag{8}$$

An immediate consequence of the preceding theorems is the following:

THEOREM 4.
$$\left.\begin{aligned} f(x)/\sim_B &= \{f(u) \mid u \in A,\ x \sim_A u\} \\ g(y)/\sim_A &= \{g(\nu) \mid \nu \in B,\ y \sim_B \nu\} \end{aligned}\right\} \quad (x \in A,\ y \in B).$$

Starting from f, g we define in the natural way the mappings (one-to-one) $F : \mathcal{P}(A) \to \mathcal{P}(B)$, $G : \mathcal{P}(B) \to \mathcal{P}(A)$.

Definition 3. $F(S) = \{f(x) \mid x \in S\} \quad (S \subseteq A)$,
$G(T) = \{g(y) \mid y \in T\} \quad (T \subseteq B)$.

THEOREM 5. *The mappings F, G have the following properties:*

(i) $F(x/\sim_A) = f(x)/\sim_B,\ G(y/\sim_A) = g(y)/\sim_A$,

(ii) *F, G are both* 1 - 1 *and onto.*

(iii) *F, G are inverse to each other.*

Proof. (i) has been proved in the preceding theorem; (ii) follows by Theorem 3 and the assumption that f, g are onto; (iii) follows by Theorem 2, part (iii).

In the theorems which follow we prove that the properties (i) - (iv) of $\sim_A$, $\sim_B$ proved in Theorem 2 and the properties (i) - (iii) of F, G proved in Theorem 5 are characteristic in the sense that any (f, g)-equivalence relation $\sim$ can be defined in terms of two equivalence relations of the sets A, B.

THEOREM 6. *Let A, B be non-empty sets, $f : A \to B$, $g : B \to A$ one-to-many mappings which are onto. Let further $\approx_A$, $\approx_B$ be binary relations of A, B respectively having the properties:*

(i) *$\approx_A$, $\approx_B$ are equivalence relations,*

(ii) *$\approx_A$, $\approx_B$ do not discern f, g respectively,*

(iii) *$x \approx_A g(f(x),\ y \approx_B f(g(y)))$, for all $f(x) \in B$, $g(y) \in A$, $x \in A$, $y \in B$*

(iv) *$\approx_A$, $\approx_B$ are compatible with f, g respectively.*

Then any (f, g)-equivalence relation $\sim$ can be defined so that the corresponding relations $\sim_A$, $\sim_B$ are just $\approx_A$, $\approx_B$.

Proof. The relation $\sim$ having the required properties is defined by:

$$x \sim y \Leftrightarrow x \approx_A g(y).$$

THEOREM 7. *Let f, g be one-to-many mappings of A onto B and B onto A respectively, and let $\approx_A$, $\approx_B$ be equivalence relations of the sets A and B. Furthermore, suppose that F, G are the mappings introduced by Definition* 1. *If F, G have the properties:*

(i) $F(x/\approx_A) = f(x)/\approx_B,\ G(y/\approx_B) = g(y)/\approx_A \quad (x \in A,\ y \in B)$,

(ii) *F, G are both* 1 - 1 *and onto,*

(iii) *F, G are inverse to each other,* *then an* (f, g)*-equivalence relation* $\sim$ *can be defined so that the corresponding relations* $\sim_A$, $\sim_B$ *are just* $\approx_A$, $\approx_B$.

The proof follows immediately by Theorems 4 and 6.

Using the preceding results and the results of [1], it follows that all (f, g)-equivalence relations can be determined by the reproductive formulae given in the next theorem.

THEOREM 8. *Let* $f : A \to B$, $g : B \to A$ *be one-to-many mappings which both are onto and* π*-inverse to each other, where* $\pi \subseteq A \times B$ *is a binary relation not discerning* f *and* g*. Then the relation* $\sim$ *defined by any of the formulae.*

$$x \sim y \Leftrightarrow (\forall z \in A)(z\pi y \Leftrightarrow z\pi f(x)) \tag{9}$$

$$x \sim y \Leftrightarrow (\forall z \in B)(x\pi z \Leftrightarrow g(y)\pi z) \tag{10}$$

is an (f, g)*-equivalence relation, and all* (f, g)*-equivalence relations can be obtained by any of the preceding formulae.*

Proof. If part: Let $\sim$ be defined by (9). It suffices to prove that $\sim$ is (f, g)-reflexive, -symmetric and -transitive. Reflexivity follows immediately. Suppose $x \sim y$, i.e. $(\forall z \in A)(z\pi y \Leftrightarrow z\pi f(x))$. As we have $s\pi y \Leftrightarrow z\pi f(g(y))$, we conclude $(\forall z \in A)(z\pi f(g(y)) \Leftrightarrow z\pi f(x))$, wherefrom $(\forall z \in A)(z\pi f(x) \Leftrightarrow z\pi f(g(y)))$, which, by definition (9), means $g(y) \sim f(x)$, i.e. $\sim$ is (f, g)-symmetric. The proof of transitivity is, for example:

$$\begin{aligned} x \sim y \land g(y) \sim z &\Rightarrow (\forall u \in A)(u\pi y \Leftrightarrow u\pi f(x)) \land (\forall u \in A)(u\pi z \Leftrightarrow u\pi f(g(y))) \\ &\Rightarrow (\forall u \in A)(u\pi y \Leftrightarrow u\pi f(x)) \land (\forall u \in A)(u\pi z \Leftrightarrow u\pi y) \\ &\Rightarrow (\forall u \in A)(u\pi z \Leftrightarrow u\pi f(x)) \\ &\Rightarrow x \sim z \end{aligned}$$

The proof of the *if part* is similar in the case $\sim$ is defined by (10). *Only if part:* By Theorem 1 parts (4) and (7), it follows immediately that each of the formulae (9), (10), is reproductive, i.e. if $\sim$ is any (f, g)-equivalence relation, it can be obtained by (9), as well as by (10), by choosing for π just the relation $\sim$.

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