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BI-QUOTIENT IMAGES OF ORDERED SPACES

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Abstract. The class of bi-quotient images of orderable spaces is characterized.

1. Introduction

In [**9**] Michael defined bi-sequential spaces as spaces in which whenever a filter base $\mathcal{F}$ accumulates at a point p (i.e. $p \in \overline{F}$ for every $F \in \mathcal{F}$) then there is a decreasing sequence $\{A_i : i \in N\}$ which meshes with $\mathcal{F}$ (i.e. every A_i intersects every $F \in \mathcal{F}$ and converges to p. He also showed that a space X is bi-sequential if and only if X is a bi-quotient image of a metrizable space [**9**] (3.D.1. and 3.D.2.). Herrlich [**5**] defined radial and pseudo-radial spaces (see [**2**], [**6**]) and proved that these spaces are exactly pseudo-open and quotient images, respectively, of ordered spaces.

In this paper we define one subclass of radial spaces as a generalization of the bi-sequential spaces; these spaces are called *biradial*. We also show (the main result) that a space is biradial if and only if it is a bi-quotient image of an ordered space.

We shall use the usual notations and terminology [**3**]. A mapping f from X onto Y is bi-quotient if whenever a filter base $\mathcal{F}$ accumulates an y in Y, then $f^{-1}(\mathcal{F})$ accumulates at some $x \in f^{-1}(y)$. Ordered space is a linearly ordered set with the interval topology. All spaces are assumed to be Hausdorff and all maps are continuous surjections.

2. Definition and characterization of biradial spaces

Definition 2.1. A space X is called biradial if whenever a filter base $\mathcal{F}$ accumulates at a point x then there is a family $\mathcal{S}$ of subsets of X so that

(i) $\mathcal{S}$ is linearly ordered by inclusion.

(ii) $\bigcap\{S : S \in \mathcal{S}\} = \{x\}$.

(iii) For any neighbourhood U of x there is an $S \in \mathcal{S}$ such that $x \in S \subset U$.

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(iv) $\mathcal{S}$ meshes with $\mathcal{F}$.

Following [**6**], we say that $\mathcal{S}$ is an r-network (c-network in [**2**] at x in X if $\mathcal{S}$ satisfies conditions (i)-(iii) of the above definition.

The following proposition is a reformulation of Definition 2.1.

PROPOSITION 2.2 *A space X is biradial if and only if whenever a filter base $\mathcal{F}$ accumulates at a point x, then there is a chain $\{x_\alpha : \alpha \in L\}$ which converges to x and every $F \in \mathcal{F}$ intersects in a cofinal subchain.*

Here "chain" means a net whose directed set is linearly ordered.

Remark 2.3. Since each linearly ordered set contains a cofinal and well-ordered subset, we may assume that L in Proposition 2.2. is well-ordered.

(Easy) *Examples* 1) Obviously, each space in which each point has a linearly ordered neighbourhood base (so-called lob-spaces or "sphérique" in [**10**]) is biradial.

In particular, every R-space in the sence of Kurepa [**7**] (i.e. a space which has a base which is a tree with respect to reverse inclusion) and every linearly uniformizable space [**4**], [**11**] (= "pseudodistanciés" [**8**] = k-metrizable [**4**]) is biradial. Let us note that R-spaces are called non-archimedean (see [**4**]).

2) All metric, all ordered and all subordered spaces are biradial.

3) Every subspace of a biradial space is biradial.

4) Every bi-sequential space is biradial. The ordinal space $[0, \omega_1]$, where ω_1 is the first uncountable ordinal, is a biradial space which is not bi-sequential.

PROPOSITION 2.4. *Every bi-quotient image of a biradial space is biradial.*

This follows by routine verification.

COROLLARY 2.5. *Every continuous image of a compact biradial space is compact biradial space.*

Remark 2.6. Biradial spaces are badly behaved with respect to products. As the product $[0, \omega_1] \times [0, \omega]$ shows, the Cartesian product of two biradial spaces is not necessarily biradial, even if both of them are compact. Let us note that every finite product of k-metrizable spaces is biradial, because every such product is k-metrizable [**11**]. Next, k-box products of at most k many k-metrizable spaces are linearly uniformizable [**4**] and thus biradial spaces.

To characterize biradial spaces as the images of ordered spaces under bi-quotient mappings, we begin with a lemma of Herrlich [**5**].

LEMMA 2.7. *If x is a point of a space X so that $Y = X\{x\}$ is disrrete and $\{x_\alpha : \alpha \in L\}$ is a well-ordered sequence such that the collection of all sets $X_\alpha = \{x\} \cup \{x_\beta : \beta > \alpha\}$, $\alpha \in L$, is a local base at x, then X is orderable.*

THEOREM 2.8 *For a space X the following conditions are equivalent:*

(1) *X i.s biradial;*

(2) *X is a bi-quotient image of an ordered space;*

(3) *X is a bi-quotient image of a topological sum of linearlv ordered spaces;*

(4) *X is a bi-quotient image of an lob-space.*

Proof. (1) $\Rightarrow$ (2) $\wedge$ (3) $\wedge$ (4). Let X be a biradial space. For each $x \in X$ and filter base $\mathcal{F}$ accumulating at x, choose a chain $C = \{x_\alpha : \alpha \in L\}$ which converges to x, such that every $F \in \mathcal{F}$ intersects in a cofinal subchain. (Without loss of generality we may assume that L is well-ordered; see Remark 2.3.) Let $Y(x, \mathcal{F}, C) = \{x^*\} \cup \{x^*_\alpha : \alpha \in L\}$ be a copy of the set $\{x\} \cup \{x_\alpha : \alpha \in L\}$, topologized so that every x^*_α is an isolated point and a base at x^* is the collection of all sets of the form $\{x^*\} \cup \{x^*_\alpha : \alpha > \beta\}$, $\beta \in L$. Let Y be the topological sum of all $Y(x, \mathcal{F}, C)$. By Lemma 2.7., Y is an orderable space (and a topological sum of orderable spaces); on the other hand, it is clear that Y is an lob-space. Let us define the natural surjection $f : Y \to X$, $f(x^*) = x$, $f(x^*_\alpha) = x_\alpha$. The map f is continuous. Clearly, it suffices to show that f is continuous at each x^*. Let V be an arbitrary neighbourhood of $f(x^*) = x$; if $C = \{x_\alpha : \alpha \in L\}$ is a chain which converges to x, then there is a $\beta \in L$ such that $x_\alpha \in V$ whenever $\alpha > \beta$, and thus $U = \{x^*\} \cup \{x^*_\alpha : \alpha > \beta\}$ is a neighbourhood of x^* for which $f(U) \subset V$. Let us show that f is bi-quotient. Suppose that $\mathcal{F}$ is a filter base accumulating at x in X; let $C = \{x_\alpha : \alpha \in L\}$ be a chain which converges to x and let every $F \in \mathcal{F}$ intersect in a cofinal subchain. Consider $Y(x, \mathcal{F}, C)$ and pick $x^a st \in f^{-1}(x)$. Obviously, every element of $f^{-1}(\mathcal{F})$ intersects every member of the local base at x^*, i.e. accumulates at x^*.

(2) $\vee$ (3) $\vee$ (4) $\Rightarrow$ (1). This follows immediately from Proposition 2.4 and the fact that every ordered and every lob-space is biradial (see Examples). This completes the proof of the theorem.

COROLLARY 2.9. *Every metrizable space (and erery lob-space) is a bi-quotient image of an ordered space.*

3. Some properties of biradial spaces

We have the following definition, analogous to Definition 6.5. in [**1**] of an absolutely Fréchet-Urysohn space:

Definition 3.1. A completely regular space X is called absolutely radial if its Stone-Cech compactification βX satisfies the following condition: for every $A \subset \beta X$ and every $x \in X \cap cl_{\beta X}(A)$ there is an r-network at x in βX which meshes with $\{A\}$.

PROPOSITION 3.2. *Every bi-quotient image of an absolutely radial space is absolutely radial.*

Proof. Let $f : X \to Y$ be a bi-quotient mapping from an absolutely radial space X onto a completely regular space Y. Let us take any subset B in βY and a point $y \in Y \cap cl_{\beta Y}(B)$. Let $\tilde{f} : \beta X \to \beta Y$ be the extension of the mapping f. By Lemma 4.2. in [**1**], we have $cl_{\beta X}(\tilde{f}^{-1}(B)) \cap f^(- 1)(y) \neq = \emptyset$, i.e. there is an $x \in X$ such that $f(x) = y$ and $x \in cl_{\beta X}(\tilde{f}^{-1}(B)) \cap X$. Since X is absolutely radial, there is an r-network $\mathcal{S}$ at x in βX which meshes with $\{A\} = \{\tilde{f}^{-1}(B)\}$. Then, as one can easily verify, $\tilde{f}(\mathcal{S})$ is an r-network at y in βY which meshes with $\{B\}$. Therefore Y is absolutely radial. Our proposition is proved.

THEOREM 3.3. *Every $T_{3\frac{1}{2}}$ biradial space is absolutely radial.*

Proof. Let A be subset of βX, $x \in X \cap cl_{\beta X}(A)$ and X biradial. We consider only the non-trivial case $x \notin A$. Let $\mathcal{U}$ be the family of all open subsets of βX such that $U \supset A$ and $x \notin U$. Put $\mathcal{F} = \{X \cap U : U \in \mathcal{U}\}$. Evidently, $\mathcal{F}$ is a filter base in X. For every $F \in \mathcal{F}$, $x \in cl_{\beta x}(F)$. Indeed, if V is any neighbourhood of x, then $V \cap A \neq \emptyset$ and thus $U \cap V \neq \emptyset$ for every $U \in \mathcal{U}$. Hence $(U \cap V) \cap X \neq \emptyset$,i.e. $V \cap (X \cap U) \neq \emptyset$. Therefore $\mathcal{F}$ accumulates at x in X. By assumption X is biradial, so there is an r-network $\mathcal{S}$ at x in X which meshes with $\mathcal{F}$. Now we claim that $\tilde{\mathcal{S}} = \{cl_{\beta X}(S) : S \in \mathcal{S}\}$ is an r-network at x in βX which meshes with $\{A\}$. Since the properties (i), (ii) and (iii) of Definition 2.1. obviously hold, we need only check that (iv) holds. We suppose that (iv) is false; then $A \cap cl_{\beta X}(S) = \emptyset$ for some $S \in \mathcal{S}$. Let $V = \beta X \setminus cl_{\beta X}(S)$. Clearly $V \in \mathcal{U}$, i.e. $V \cap X \in \mathcal{F}$; thus $S \cap (V \cap X) \neq \emptyset$, whirc is a contradiction. This proves that X is absolutely radial and Theorem 3.3. is proved.

It is natural to ask when a biradial space is bi-sequential. The proof of the following theorem is similar to the proof of Theorem 3 in [**6**] which states that every pseudo-radial space of countable pseudocharacter is sequential.

THEOREM 3.4. *Every biradial space of countable pseudocharacter is bi-sequential.*

Proof. Let X be a biradial space of countable pseudocharacter, and a filter base accumulating at a point x. Let $\{U_i : i \in N\}$ be a family of open subsets of X such that $\bigcap\{U_i : i \in N\} = \{x\}$. Since X is a biradial space there exists an r-network $\mathcal{S}$ at x which meshes with $\mathcal{F}$. We may suppose that $x \notin F_0$ for some $F_0\mathcal{F}$ (if $x \in \bigcap\{F : F \in \mathcal{F}\}$ the proof is trivial). For each $i \in N$ let S_i be an element of $\mathcal{S}$ such that $x \in S_i \subset U_i$. We claim that $\tilde{\mathcal{S}} = \{S_i : i \in N\}$ is an r-network at x which meshes with $\mathcal{F}$. Clearly, we need only prove that (iii) in Definition 2.1. holds, since, obviously, all the conditions (i), (ii) and (iv) hold. Let us suppose that (iii) is not true. Then there exists a neighbourhood V of x such that $S_i \setminus V \neq \emptyset$ for every $i \in N$. On the other hand, there is an $S^* \in \mathcal{S}$ such that $x \in S^* \subset V$. Since $\mathcal{S}$ is linearly ordered we have: $S^* \subset \bigcap\{S_i : i \in N\} \subset \{U_i : i \in N\} = \{x\}$. But, $S^* \cap F_0 \neq \emptyset$, and thus $S^* \cap (X \setminus \{x\}) \neq \emptyset$, which is a contradiction. Therefore the claim is proved. In other words: there is a countable filter base $\mathcal{S}$ wich meshes with $\mathcal{F}$ and converges to x. Thus X is a bi-sequential space. This completes the proof.

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