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CERTAIN APPLICATIONS OF DIFFERENTIAL SUBORDINATION

K. S. Padmanabhan and R. Manjini

Abstract. Let A denote the class of functions f regular in the unit disc E, such that $f(0) = 0 = f'(0) - 1$. Let $k_a(z) = z/(1-z)^a$ where a is a real number. We denote by $K_a(h)$ the class of functions $f \in A$ satisfying $1 + \frac{z(k_a * f)''(z)}{(k_a * f)'(z)} \prec h(z)$, where h is a convex univalent function in E with $h(0) = 1$ and $\mathrm{Re}\,(h(z)) > 0$. Several properties of the class $K_a(h)$ are investigated. Certain allied classes are also studied.

Let $E = \{z \in C : |z| < 1\}$ and $H(E)$ be the set of all functions holomorphic in E. Let $A = \{f \in H(E) : f(0) = 0 = f'(0) - 1\}$. By $f * g$ we denote the Hadarnard product or convolution of $f, g \in H(E)$. That is, if $f(z) = \sum_0^\infty a_n z^n$, $g(z) = \sum_0^\infty b_n z^n$, then $(f * g)(z) = \sum_0^\infty a_n b_n z^n$.

Let g and G be two functions in $H(E)$. Then we say that $g(z)$ is subordinate to $G(z)$ (written $g(z) \prec G(z)$) if $G(z)$ is univalent, $g(0) = G(0)$ and $g(E) \subset G(E)$. Let $k_a(z) = z/(1-z)^a$, where a is any real number. From now on we assume, unless otherwise stated, $h \in H(E)$ is convex univalent in E and satisfies $h(0) = 1$ and $\mathrm{Re}\,(h(z)) > 0$ for $z \in E$.

Definition A. [**2**]. An infinite sequence $\{d_n\}_1^\infty$ of complex numbers is said to preserve property T if whenever $f(z) = \sum_1^\infty a_n z^n$ possesses property T, the convolution $J(z) = f(z) * \sum_1^\infty d_n z^n$ also possesses property T.

Definition B. [**8**]. Let $S_a(h)$ denote the class of functions $f \in A$ such that $\frac{z(k_a * f)'(z)}{k_a * f)(z)} \prec h(z)$, where $(k_a * f)(z)/z \neq 0$, for $z \in E$.

Definition C. [**8**]. Let $C_a(h)$ denote the class of functions $f \in A$ such that $\frac{z(k_a * f)'(z)}{k_a \varphi)(z)} \prec h(z)$, for some $\varphi \in S_a(h)$.

When $a = 1$ and $h(z) = (1+z)/(1-z)$, the classes $S_a(h)$, $C_a(h)$ reduce to the familiar classes S^* (starlike univalent functions), C (close-to-convex functions) respectively. We need the following five lemmas in the sequel.

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LEMMAA. [**3**]. *Let $\beta, \gamma \in C$, let $h \in H(E)$ be convex uniralent in E with $h(0) - 1$ and $\operatorname{Re}(\beta h(z) + \gamma) > 0$, $z \in E$, and let $p \in H(E)$, $p(z) = 1 + p_1 z + \ldots$. Then*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z)$$

implies that $p(z) \prec h(z)$.

LEMMA B. [**8**]. *Suppose $f \in S_a(h)$ and*

$$F(z) = \frac{\gamma+1}{z^\gamma} \int_0^x t^{\gamma-1} f(t) dt = \sum_{n=1}^\infty \left(\frac{\gamma+1}{\gamma+n}\right) a_n z^n. \tag{1}$$

where $\operatorname{Re}\gamma > 0$ and $f(z) = z + \sum_2^\infty a_n z^n$. Then $F \in S_a(h)$, provided $(k_a * F)(z)/z \neq 0$ for $z \in E$.

LEMMA C. [**8**] *Suppose $f \in C_a(h)$ with respect to the function $\varphi \in S_a(h)$. Define φ by $\varphi(z) = (\varphi * h_\gamma)(z)$, where $h_\gamma(z) = \displaystyle\sum_{n=1}^\infty \left(\frac{\gamma+1}{\gamma+n}\right) z^n$. Then $F(z)$, defined by (1), is in $C_a(h)$ with respect to φ provided $(k_a * \varphi)z/z \neq 0$ for $z \in E$.*

LEMMA D. [**4** p. 12]. *Suppose that $h(z) = \sum_2^\infty h_n z^n$ is convex univalent and maps $|z| < 1$ onto D. Let $\omega = g(z) = \sum_1^\infty g_n z^n$ be regular in $|z| < 1$ and assume there are only values ω which lie in D. Then $|g_n| \leq |h_1|$ and in particular $|h_n| \leq |h_1|$ for $n \geq 1$.*

LEMMA E. [**1**]. *Let $\varphi \in K$, the class of convex univalent functions, $g \in S^*$ and $F \in H(E)$ such that $\operatorname{Re} F > 0$. Then $(\varphi * Fg)/(\varphi * g)$ lies in the convex hull of $F(E)$.*

For $F \in A$ and $k_a(z) = z/(1-z)^a$, a is any real number, we have the following easily verified result:

$$z(k_a * f)'(z) = a(k_{a+1} * f)(z) - (a-1)(k_a * f)(z). \tag{2}$$

Definition 1. Let $K_a(h)$ denote the class of functions $f \in A$ such that

$$1 + \frac{z(k_a * f)''(z)}{(k_a * f)'(z)} \prec h(z), \quad \text{whwre } (k_a * f)'(z) \neq 0 \text{ for } z \in E.$$

Remark 1. If $a = 1$ and $h(z) - (1+z)/(1-z)$, then $K_a(h) = K$, the class of convex univalent functions.

THEOREM 1. *If $f \in K_{a+1}(h)$, then $f \in K_a(h)$ for $a \geq 1$.*

Proof. Let $p(z) = \dfrac{z(k_a * f)''(z)}{(k_a * f)'(z)}$. Differentiating (2), we get

$$p(z) + (a-1) = a\frac{(k_{a+1} * f)'(z)}{(k_a * f)'(z)}.$$

Taking logarithmic derivatives and multiplying by z, we get

$$\frac{zp'(z)}{p(z)+(a-1)} = \frac{z(k_{a+1}*f)''(z)}{(k_{a+1}*f)'(z)} - \frac{z(k_a*f)''(z)}{(k_a*f)'(z)},$$

which gives

$$1+\frac{z(k_{a+1}*f)''(z)}{k_{a+1}*f)'(z)} = \frac{zp'(z)}{p(z)+(a-1)} + p(z). \tag{3}$$

If $f \in K_{a+1}(h)$, from (3) we have

$$\frac{zp'(z)}{p(z)+(a-1)} + p(z) \prec h(z).$$

From Lemma A it follows that $p(z) \prec h(z)$ for $a \geq 1$; that is, $1+\dfrac{z(k_a*f)''(z)}{(k_a*f)'(z)} \prec h)z)$, which means $f \in K_a(h)$ for all $a \geq 1$.

THEOREM 2. *Suppose $f \in K_a(h)$ and F is defined by* (1). *Then $F \in K_a(h)$ provided $(k_a*F)'(z) \neq 0$ for $z \in E$.*

Proof. We have $zF'(z) + \gamma F(z) = (\gamma+1)f(z)$ and so

$$(k_a*(zF'))(z) + \gamma(k_a*F)(z) = (\gamma+1)(k_a*f)(z).$$

Using the fact

$$z(k_a*F)'(z) = (k_a*zF')(z), \tag{4}$$

we obtain

$$z(k_a*F)'(z) + \gamma(k_a*F)(z) = (\gamma+1)(k_a*f)(z). \tag{5}$$

Let $p(z) = 1+\dfrac{z(k_a*F)''(z)}{(k_a*F)'(z)}$. Differentiating (5), we get

$$p(z)+\gamma = (\gamma+1)\frac{(k_a*f)'(z)}{(k_a*F)'(z)}$$

and so we have

$$\frac{zp'(z)}{p(z)+\gamma} + p(z) = 1+\frac{z(k_a*f)''(z)}{(k_a*f)'(z)}. \tag{6}$$

We conclude, if $f \in K_a(h)$, from (6) and Lemma A that $p(z) \prec h(z)$. Thus $F \in K_a(h)$.

COROLLARY 2.1. *For every γ with $\operatorname{Re}\gamma > 0$, the sequence $\{(\gamma+1)/(\gamma+n)\}$ preserves the property $f \in K_a(h)$.*

Proof. Corollary follows from Definition A with $d_n = (\gamma+1)/(\gamma+n)$.

Remark 2. If $a = 1$ and $h(z) = (1+z)/(1-z)$, we deduce Theorem 2 and Corollary 2.1 of Bernardi [**2**] from the above theorem and its corollary.

THEOREM 3. (i) *$f \in K_a(h)$ if and only if $zf' \in S_a(h)$.* (ii) *If $f \in K_a(h)$, then $f \in S_a(h)$; that is, $K_a(h) \subset S_a(h)$.*

Proof. Using (4) we find that

$$\frac{z(k_a * zf)'(z)}{(k_a * zf')(z)} = 1 + \frac{z(k_a * f)''(z)}{(k_a * f)'(z)},$$

which implies (i).

Let $p(z) = \dfrac{z(k_a * f)'(z)}{(k_a * f)(z)}$. Then

$$p(z) + \frac{zp'(z)}{p(z)} = 1 + \frac{z(k_a * f)''(z)}{(k_a * f)'(z)}. \tag{7}$$

If $f \in K_a(h)$, then $f \in S_a(h)$ by (7) and Lemma A.

Remark 3. If $a = 1$ and $h(z) = (1+z)/(1-z)$, then part (i) reduces to the well-known resul that zf' is starlike if and only if f is convex, and part (ii) reduces to the well-known result that the class of convex univalent functions is contained in the class of starlike univalent functions.

COROLLARY 3.1. *If $f \in S_a(h)$, then $\displaystyle\int_0^z \frac{\gamma+1}{\gamma+n}\left[\int_0^t x^{\gamma-1} f(x)dx\right] dt$ is in $K_a(h)$.*

Proof. Let $f \in S_a(h)$. Then $\dfrac{\gamma+1}{\gamma^z}\displaystyle\int_0^z x^{\gamma-1} f(x)dx$ is in $S_a(h)$ by Lemma B. By part (i) of Theorem 3 there is a function $g \in K_a(h)$ such that $zg'(z) = \dfrac{\gamma+1}{\gamma^z}\displaystyle\int_0^z x^{\gamma-1} f(x)dx$ which implies the result of the corollary.

COROLLARY 3.2. *If $f \in K_a(h)$ and $h(z) = (\gamma+1)f(z) - \gamma F(z)$, where F is defined by* (1), *then $h \in S_a(h)$.*

Proof. Let $f \in K_a(h)$. Then by Theorem 2 we have $F \in K_a(h)$. By part (i) of Theorem 3, $zF' \in S_a(h)$. But $zF'(z) = (\gamma+1)f(z) - \gamma F(z)$. Hence the corollary.

THEOREM 4. *Let $\varphi \in K$, $g \in S_a(h)$. Then $\varphi * g \in S_a(h)$.*

Proof. Let $F = \dfrac{z(k_a * g)'(z)}{(k_a * g)(z)}$ so that $F \prec h$. Now

$$\frac{z(k_a * \varphi * g)'(z)}{(k_a * \varphi * g)(z)} = \frac{z(\varphi * (k_a * g))'(z)}{(\varphi * (k_a * g))(z)} = \frac{(\varphi * z(k_a * g)')(z)}{(\varphi * (k_a * g))(z)} = \frac{(\varphi * F(k_a * g))(z)}{(\varphi * (k_a * g))(z)}$$

Since $g \in S_a(h)$, $k_a * g \in S^*$ and it follows from Lemma E that $z(k_a * \varphi * g)'(z)/(k_a * \varphi * g)(z)$ lies in the convex hull of $F(E)$. But $F \prec h$, where h is convex. So the convex hull of $F(E)$ is a subset of $h(E)$ and the conclusion follows.

COROLLARY 4.1. *Let $\varphi \in K$, $f \in K_a(h)$. Then $\varphi * f \in K_a(h)$.*

Proof. By Theorem 3, $f \in K_a(h)$ if and only if $zf' \in S_a(h)$. $z(\varphi * f)'(z) = (\varphi * zf')(z) \in S_a(h)$ by Theorem 4. Hence $\varphi * f \in K_a(h)$.

THEOREM 5. *Let $f \in A$ and let h be continuous on the unit circle, besides ,satisfying the usual conditions. $f \in S_a(h)$ if and only if $(k * f)(z) \neq 0$, $z \neq 0$, and*

$$f(z) * \frac{z[1 - h(x) + (a + h(x))z]}{(1 - z)^{a+1}} \neq 0, \quad 0 < |z| < 1, \ |x| = 1. \tag{8}$$

Proof. Let $f \in A$ satisfy $(k_a * f)(z) \neq 0$, $z \neq 0$ and (8). Put $g(z) = (k_a * f)(z)$. Then $g(z) \neq 0$ for $0 < |z| < 1$. We can rewrite (8) as

$$G(z) = \frac{(k_{a+1} * f)(z)}{(k_a * f)(z)} \neq \frac{a-1}{a} + \frac{1}{a}h(x), \quad |x| = 1, \ z \in E. \tag{9}$$

From (2) we get

$$G(z) = \frac{a-1}{a} + \frac{1}{a}\frac{zg'(z)}{g(z)}, \quad z \in E. \tag{10}$$

(9) and (10) imply $zg'(z)/g(z) \neq h(x)$, $|x| = 1$, $z \in E$. $zg'(z)/g(z)|^{z=0} = 1 \in h(E)$. Also $zg'(z)/g(z)$ is analytic in E and so maps E onto a region which conttains 1 and is a subset of $h(E)$. Therefore $zg'(z)/g(z) \prec h(z)$. Hence $f \in S_a(h)$.

Conversely, $f \in S_a(h)$ implies $zg'(z)/g(z) \prec h(z)$, $z \in E$ and so $zg'(z)/g(z) \neq h(x)$, $|x| = 1$, $z \in E$. By retracing the steps we obtain the converse.

Definition 2. Let $K_a^\alpha(h)$, α be any real number, denote the class of functions $f \in A$ such that

$$J_a(\alpha; f(z)) = \alpha\left(1 + \frac{z(k_a * f)''(z)}{(k_a * f)'(z)}\right) + (1 - \alpha)\frac{z(k_a * f)'z}{(k_a * f)(z)} \prec h(z)$$

with $(k_a * f)(z)/z \neq 0$ and $(k_a * f)'(z) \neq 0$ for $z \in E$.

Remark 4. When $a = 1$ and $h(z) = (1 + z)/(1 - z)$, $K_a^\alpha(h)$ is the class of all α-convex functions introduced by Mocanu [**6**].

For $\alpha = 1$, the class $K_a^\alpha(h)$ coincides with the class $K_a(h)$; and for $\alpha = 0$, it reduces to the class $S_a(h)$. Thus the sets $K_a^\alpha(h)$ give a "continuous" passage from the class $K_a(h)$ to the class $S_a(h)$.

THEOREM 6. (i) *If $f \in K_a^\alpha(h)$, then $f \in K_a^0(h) = S_a(h)$ for $\alpha > 0$.* (ii) *For $\alpha > \beta \geq 0$. $K_a^\alpha(h) \subset K_a^\beta(h)$.*

Proof. (i) Let $p(z) = \frac{z(k_a * f)'(z)}{(k_a * f)(z)}$. Then, using (7), we find that $J_a(\alpha; f(z)) = \alpha z p'(z) + p(z)$. If $f \in K_a^\alpha(h)$, then, by Lemma A, we have $p(z) \prec h(z)$ if $\alpha > 0$. That is, $f \in K_a^0(h) = S_a(h)$ for $\alpha > 0$.

(ii) If $\beta = 0$, then this statement reduces to (i). Hence we assume that $\beta \neq 0$. Suppose $f \in K_a^\alpha(h)$. Then $J_a(\alpha; f(z)) \prec h(z)$. Let z_1 be arbitrary point in E. Then

$$J_a(\alpha; f(z_1)) \in H(E). \tag{11}$$

Also, by part (i) $\dfrac{z(k_a * f)'(z)}{(k_a * f)(z_1)} \in h(z)$; so we have

$$\frac{z_1(k * f)'(z_1)}{(k_a * f)(z_1)} \in H(E). \tag{12}$$

Now

$$J_a(\beta; f(z)) = \left(1 - \frac{\beta}{\alpha}\right) \frac{z(k_a * f)'(z)}{(k_a * f)(z)} + \frac{\beta}{\alpha} J_a(\alpha; f(z)).$$

Since $\beta/\alpha < 1$ and $h(E)$ is convex, $J_a(\beta; f(z_1)) \in H(E)$ by (11) and (12). Therefore $J_a(\beta; f(z)) \prec h(z)$. That is, $f \in K_a^\beta(h)$.

Remark 5. If $a = 1$ and $h(z) = (1+z)/(1-z)$, then the first part of Theorem 6 reduces to the result due to Mocanu and Reade [**7**] that all α-convex functions are starlike and the second part of Theorem 6 reduces to a result of Sakaguehi [**9**].

THEOREM 7. (i) *If* $f \in K_a^\alpha(h)$, $F(z) = (k_a * f)(z) \left[\dfrac{z(k_a * f)'(z)}{(k_a * f)(z)}\right]^\alpha$, *and if we choose that branch of* $\left[\dfrac{z(k_a * f)'(z)}{(k_a * f)(z)}\right]^\alpha$ *which is equal to* 1 *at* $z = 0$, *then* $F \in S_1(h)$.

(ii) *If* $F(z) = f \int_0^z [(k_a * f)(t)/t]^{1-\alpha} ((k_a * f)'(t))^\alpha dt$, *then* $F \in K_1(h)$ *if and only if* $f \in K_a^\alpha(h)$.

Proof. (i) From the definition, we have $F(0) = 0$, $F'(0) = 1$, and

$$zF'(z)/F(z) = J_a(\alpha; f'(z)) \prec h(z),$$

since $f \in K_a^\alpha(h)$. So $F \in S_1(h)$.

(ii) From the definition of F, we have

$$F'(z) = [(k_a * f)(z)/z]^{1-\alpha} ((k_a * f)'(z))^\alpha$$

and so $1 + zF''(z)/F'(z) = J_a(\alpha; f'(z))$. Hence $F \in K_1(h)$ if and only if $f \in K_a^\alpha(h)$.

Remark 6. If $a = 1$ and $h(z) = (1+z)/(1-z)$, then part (i) reduces to a result of Mocanu [**6**] and part (ii) reduces to a result of Umezawa and Takijama [**10**].

Definition 3. Let $B_a(\alpha)$, $\alpha > 0$, be the class of functions $f \in A$ such that

$$f(z) = \left[\alpha \int_0^z (k_a * g)^{\alpha}(t) \frac{dt}{t}\right]^{1/\alpha}, \quad \text{where } g \in S_a(h).$$

THEOREM 8. *If $f \in B_a(1/\alpha)$, $\alpha > 0$, then $f \in K_1^{\alpha}(h)$.*

Proof. Let $f \in B_a(1/\alpha)$. Then $f(z) = \left[\frac{1}{\alpha} \int_0^z (k_a * g)^{1/\alpha}(t) \frac{dt}{t}\right]^{\alpha}$; so

$$J_1(\alpha; f(z)) = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f(z)}\right) = \frac{z(k_a * g)'(z)}{(k_a * g)(z)} \prec h(z),$$

since $g \in S_a(h)$. Hence $f \in K_1^{\alpha}(h)$.

Remark 7. If $a = 1$ and $h(z) = (1+z)/(1-z)$, then the class $B_a(\alpha) = B(\alpha)$, the class of all Bazilevic functions of type a and Theorem 8 reduces to Theorem 1 of Miller, Mocanu and Reade [**5**].

Definition 4. If $f(z) \in S_a(h)$ and $\alpha = \alpha(f) = 1.\text{u.b.}\ [\beta/f \in K_a^{\beta}(h), \beta \geq 0]$, then we say that $f(z)$ is of type a in $S_a(h)$, and we write $f \in K(a, \alpha)$. We note that α is non-negative and may be infinite.

THEOREM 9. (i) *$f \in K(a, \alpha)$ for $\alpha < \infty$ if and only if $f \in K_a^{\beta}(h)$ for all β, $0 \leq \beta \leq \alpha$ and $f \notin K_a^{\alpha}(h)$ for $\beta > \alpha$.*

(ii) $S_a(h) = \bigcup_{\alpha \geq 0} K(a, \alpha)$, *the sets $K(a, \alpha)$, $\alpha \geq 0$ being disjoint.*

Proof. (i) If $f \in K(a, \alpha)$, then $J_a(\beta; f(z)) \prec h(z)$ holds for $z \in E$ and for all β, $0 < \beta < \alpha$. So $f \in K_a^{\beta}(h)$ for $0 \leq \beta < \alpha$. By letting $\beta \to \alpha$ we note that $J_a(\alpha; f(z))$ lies in $\overline{h(E)}$ for all $z \in E$, where $\overline{h(E)}$ is the closure of $H(E)$. Since $J_a(\alpha; f(z))$ is an analytic function in E, by open mapping theorem, the image of E by $J_G(\alpha; f(z))$ must be a region or a point. But $J_a(\alpha; f(z))$ is not a constant function because $f(z)$ is not constant. Therefore, the range of $J_a(\alpha; f(z))$ must be a region and so $J_a(\alpha; f(z))$ lies in $h(E)$ for all $z \in E$. That is, $J_a(\alpha; f(z)) \prec h(z)$. Hence $f \in K_a^{\alpha}(h)$.

The converse follows from the definition of $K(a, \alpha)$.

(ii) From the definition, we can write $S_a(h) = \bigcup_{\alpha \geq 0} K(a, \alpha)$. Since, by part (i), $K(a, \alpha) \neq K(a, \beta)$ if $\alpha \neq \beta$, the union is disjoint.

Example. Let $f(z) \equiv z$. From the definition of $J_a(\alpha; f(z))$ we find that $J_a(\alpha; z) \equiv 1$. Hence $J_a(\alpha, z) \prec h(z)$ for all $\alpha > 0$; that is, $f \in K_a^{\alpha}(h)$ for all $\alpha > 0$ and hence $f \in K(\alpha, \infty)$.

THEOREM 10. *If $f \in K(a, \alpha)$, $\alpha > 0$, and if for $0 < \beta \leq \alpha$, we choose the branch of $\left[\frac{z(k_a * f)'(z)}{(k_a * f)(z)}\right]^{\beta}$ which is equalto 1 when $z = 0$, then the function*

$$F_{\beta}(z) = (k_a st f)(z)\left[\frac{z(k_a * f)'(z)}{(k_a * f)(z)}\right]^{\beta}, \quad 0 \leq \beta \leq \alpha,$$

is in $S_1(h)$ *for all* β, $0 \le \beta \le \alpha$.

Proof. If $f \in K(a,\alpha)$, then, by part (i) of Theorem 9, we have $f \in K_a^\beta(h)$ for all β, $0 \le \beta \le \alpha$. By part (i) of Theorem 7, we have $F_\beta(z) \in S_1(h)$ for all β, $0 \le \beta \le \alpha$.

Remark 8. If $a = 1$ and $h(z) = (1+z)/(1-z)$, then Theorem 10 reduces to Theorem 4 and Theorem 9 reduces to the remark before Theorem 4 of Miller, Mocanu and Reade [**5**].

Definition 5. Let $P_a(h)$ denote the class of functions $f \in A$ such that $(k_a * f)'(z) \prec h(z)$, for $z \in E$.

THEOREM 11. (i) *If* $f \in P_{a+1}(h)$, *then* $f \in P_a(h)$ *holds for* $a > 0$.

(ii) *If* $f \in P_a(h)$ *then* $F \in P_a(h)$, *where* F *is defined by* (1).

Proof. (i) Let $p(z) = (k_a * f)'(z)$. Then, by (2), we have

$$zp(z) = a(k_{a+1} * f)(z) - (a-1)(k_a * f)(z);$$

and so

$$zp'(z)/a + p(z) = (k_{a+1} * f)'(z). \tag{13}$$

If $f \in P_{a+1}(h)$, then from (13) and Lemma A, it follows that for $a > 0$, $p(z) \prec h(z)$. That is, $f \in P_a(h)$ for all $a > 0$.

(ii) Let $p(z) = (k_a * F)'(z)$. From (5) we have

$$zp(z) + \gamma(k_a * F)(z) = (\gamma+1)(k_a * f)(z). \tag{14}$$

Differentiating (14), we get

$$zp'(z)/(\gamma+1) + p(z) = (k_a * f)'(z) \prec h(z),$$

since $f \in P_a(h)$. Then $F \in P_a(h)$ follows from lemma A.

Remark 9. If $a = 1$ and $h(z) = (1+z)/(1-z)$, then $P_a(h)$ is the class of functions whose derivatives have a positive real part and part (ii) of Theorem 11 reduces to Theorem 4 of Bernardi [**2**].

Definition 6. Let $P_a^\alpha(h)$, $\alpha > 0$, denote the class of functions $f \in A$ such that $\alpha(k_{a+1} * f)'(z) + (1-\alpha)(k_a * f)'(z) \prec h(z)$ for $z \in E$.

THEOREM 12. (i) *If* $f \in P_a^\alpha(h)$, *then* $f \in P_0^a(h) = P_a(h)$, *for* $a > 0$.

(ii) *For* $\alpha > \beta \ge 0$ *and* $a > 0$, $P_a^\alpha(h) \subset P_a^\beta(h)$.

Proof. (i) Let $p(z) = (k_a * f)'(z)$. By (13), we have

$$\alpha(k_{a+1} * f)'(z) + (1-a)(k_a * f)'(z) = \alpha zp'(z)/a + p(z).$$

If $f \in P_a^\alpha(h)$, then $\alpha zp'(z)/a + p(z) \prec h(z)$. By Lemma A, $f \in P_a(h)$ for $a > 0$.

(ii) Proof of this part is similar to that of part (ii) of Theorem 6.

Definition 7. Let $R_a(h)$ denote the class of functions $f \in A$ such that $(k_a * f)(z)/z \prec h(z)$, for $z \in E$.

Remark 10. If $a = 1$ and $h(z) = (1+z)/(1-z)$, then $R_a(h)$ is the class of functions such that $\mathrm{Re}\,(f(z)/z) > 0$.

THEOREM 13. (1) *If $f \in R_{a+1}(h)$, then $f \in R_a(h)$ for $a > 0$* (ii) *If $f \in R_a(h)$, then $F \in R_a(h)$, where F is defined by* (1).

Proof. (i) Let $p(z) = (k_a * f)(z)/z$. Then we have

$$zp'(z) + p(z) = (k_a * f)'(z). \tag{15}$$

By (2) and (15),

$$zp'(z)/a + p(z) = (k_a * f)(z)/z. \tag{16}$$

By Lemma A and (16) we conclude that $f \in R_a(h)$ for $a > 0$ if $f \in R_{a+1}(h)$.

(ii) Let $p(z) = (k_a * F)(z)/z$. Then $zp'(z) + p(z) = (k_a * F)'(z)$. Using (5) we get

$$zp'(z)/(\gamma+1) + p(z) = (k_a * f)(z)/z \prec h(z)$$

if $f \in R_a(h)$. By Lemma A, it follows that $F \in R_a(h)$.

THEOREM 14. (i) *$f \in P_a(h)$ if and only if $zf' \in R_a(h)$.* (ii) *Let $a > 0$. Then $f \in P_a^a(h)$ if and only if $zf' \in P_a(h)$.*

Proof. (i) $(k_a * zf')(z)/z = (k_a * f)'(z)$. This implies part (i).

(ii) From (2), we have

$$(k_a * zf')(z) = a(k_{a+1} * f)(z) - (a-1)(k_a * f)(z).$$

Differentiating the above equation, we get

$$(k_a * zf')'(z) = a(k_{a+1} * f)'(z) + (1-a)(k_a * f)'(z).$$

From the above equation we get part (ii).

Definition 8. Let $R_a^\alpha(h)$, $\alpha > 0$, denote the class of function $f \in A$ such that

$$\alpha(k_{a+1} * f)(z)/z + (1-a)(k_a * f)(z)/z \prec h(z), \quad \textit{for}\ \ z \in E.$$

THEOREM 15. (i) *If $f \in R_a^\alpha(h)$, then $f \in R_0^a(h) = R_a(h)$, for $a > 0$.*

(ii) *For $\alpha > \beta \geq 0$ and $a > 0$, $R_a^\alpha(h) \subset R_a^\beta(h)$.*

Proof. Proof of this theorem is similar to that of Theorem 12.

THEOREM 16. (i) *The sets $P_a(h)$ and $R_a(h)$ are convex.* (ii) *If $f \in P_a(h)$, then $\left|\binom{a+n-2}{n-1}a_n\right| \leq |h_1|/n$, $n = 2, 3, \ldots$.* (iii) *If $f \in R_a(h)$, then $\left|\binom{a+n-2}{n-1}a_n\right| \leq |h_1|$, $n = 2, 3, \ldots$, where $h(z)$ is of the form $h(z) = 1 + \sum_1^\infty h_n z^n$, $f(z) = z + \sum_1^\infty a_n z^n$ and $\binom{a}{n} = \dfrac{a(a-1)(a-2)\ldots(a-n+1)}{1\cdot 2\cdot 3\ldots(n-1)n}$.*

Proof. (i) Let f and g be in $P_a(h)$. Then $(k_a * f)'(z) \prec h(z)$ and $(k_a * g)'(z) \prec h(z)$. Let z_1 be arbitrary point in E. Then $(k * f)'(z_1) \in H(E)$ and $k_a * g)'(z_1) \in H(E)$. Since $h(E)$ is convex for $0 \leq t \leq 1$, we have

$$t(k_a * f)'(z_1) + (1-t)(k_a * g)'(z_1) \in H(E);$$

that is, $[k_a * (tf - (1-t)g)]'(z_1) \in H(E)$. Therefore $[k_a * (tf + (1-t)g)'(z) \prec h(z)$, which implies $tf + (1-t)g \in P_a(h)$. Thus $P_a(h)$ is convex. Similarly we can prove $R_a(h)$ is convex:

(ii) $(k_a * f)(z) = \sum_2^\infty \binom{a+n-2}{n-1} a_n z^n$

and so

$$(k_a * f)'(z) = 1 + \sum_2^\infty n \binom{a+n-2}{n-1} a_n z^{n-1}.$$

If $f \in P_a(h)$, then $(k_a * f)' \prec h(z)$, which implies

$$\sum_2^\infty n \binom{a+n-2}{n-1} a_n z^{n-1} \prec \sum h_a z^n.$$

By Lemma D we have the result. Part (iii) can be proved in a similar way.

Definition 9. Let $f(z) = z + \sum_2^\infty a_n z^n$ be in A. Define

$$F_p(z) = \sum_{n=1}^\infty \left(\frac{1+\gamma_1}{n+\gamma_1} \cdot \frac{1+\gamma_2}{n+\gamma_2} \cdots \frac{1+\gamma_p}{n+\gamma_p} \right) a_n z^n,$$

$$F_{p+1}(z) = \sum_{n=1}^\infty \left(\frac{1+\gamma_1}{n+\gamma_1} \cdot \frac{1+\gamma_2}{n+\gamma_2} \cdots \frac{1+\gamma_p}{n+\gamma_p} \right) \left(\frac{1+\gamma_{p+1}}{n+\gamma_{p+1}} \right) a_n z^n,$$

where $p = 1, 2, 3, \ldots$, $\operatorname{Re}\gamma_p > 0$ and $F_0(z) \equiv f(z)$. Let $g(z) = z \sum_2^\infty d_n z^n$, $G_p(z)$, $G_{p+1}(z)$ be similarly defined with identical γ_i as in $F_p(z)$ and $F_{p+1}(z)$ but with d_n in place of a_n. (The γ_i may or may not be distinct.)

Theorem 17. *Let $f(z)$, $g(z)$, $F_p(z)$, $F_{p+1}(z)$, $G_p(z)$, $G_{p+1}(z)$ be defined as in Definition 9. Then for $p = 1, 2, 3, \ldots$, we have $F_p \in S_a(h)$, $K_a(h)$, $P_a(h)$, $R_a(h)$, according to whether $f \in S_a(h)$ $K_a(h)$, $P_a(h)$ or $R_a(h)$ respectively. Also if $f(z) \in C_a(h)$ with respect to $G_p(z) \in S_a(h)$.*

Proof. From the definition of $F(z)$ we have the following recursive relations

$$F_{p+1}(z) = (1+\gamma_{p+1}) z^{-\gamma_{p+1}} \int_0^z t^{-1+\gamma_{p+1}} F_p t(dt).$$

We also have similar relation for $G_p(z)$. The results follow respectively from Lemma B, Theorem 2, Theorem 11, Theorem 13, and Lemma C, together with the above recursive relations.

Remark 11. If $a = 1$ and $h(z) = (1+z)(1-z)$, then this theorem reduces to Theorem 5 of Bernardi [**2**].

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The Ramanujan Institute,
University of Madras,
Madras – 600 005,
India.

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