PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 39 (53), 1986, pp. 97–106

CHARACTERIZATION OF A FULL SET OF PROBABILITIES ON SOME POSETS

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Abstract. We define algebraic structures which contain all Boolean algebras pseudo-Boolean algebras, and orthocomplenented posets, as special cases; then we introduce the notion of orthonogality and, after defining probability in the usual way, we characterize those among our new structures that admit a full set of probabilities. Some special cases of the main result about characterization are considered and, especially, a result concerning orthocomplemented posets is derived from it. Also, sufficient conditions for the existence of a full set of probabilities on a pseudo-Boolean poset is obtained.

1. Introduction. Our intention in this section it to define algebraic structures, we will call quasi-complemented posets, and to clarify their relationship with orthocomplemented posets and pseudo-Boolean algebras (for definitions of the latter two structures see e. g. [**1**] and [**8**], respectively). Let S be an arbitrary poset with minimal and maximal element (denoted by 0 and 1, respectively) and let Q be a mapping from $S \times S$ into S, assigning $a^b \in S$ to every pair $(a, b) \in S \times S$.

(1.1) *Definition.* A mapping Q is called relative quasi-complementation on S if it satisfies the following conditions

(1.1.1) if $c \in S$ is such that $c \leq a$ and $c \leq a^b$, then $c \leq b$;

(1.1.2) $a \leq (a^0)^0$ for all $a \in S$;

(.1.1.3) if $a \leq b$, then $b^c \leq a^c$ for all $a, b, c \in S$.

A mapping Q is called relative $*$-quasi-complementation if, in addition, the following condition is satisfied:

(1.1.4) if $s \in S$ is such that for all $c \in S$ relations $c \leq a$ and $c \leq s$ imply $c \leq b$, then either $s \leq a^b$ or s and a^b are not comparable.

Although conditions (1.1.2) and (1.1.3) reflect well our intuition about complementation, it is not immediately clear when a mapping with properties (1.1.1)-(1.1.3), that is (1.1.1)-(1.1.4), exists. To establish that, let us denote by $Q_{(a,b)}$, for

AMS Subject Classification (1984): Primary 28C99, Secondary 06A99 given $a, b \in S$, the set of all elements from S which satisfy (1.1.1) and by $Q^*_{(a,b)}$ the set of all elements from S which, in addition, satisfy (1.1.4). It is clear that, although the set $Q^*_{(a,b)}$ need not exist, the set $Q_{(a,b)}$ is always non-empty; obviously $Q^*_{(a,b)} \subseteq Q_{(a,b)}$. We shall consider only those posets for which $Q^*_{(a,b)} \neq \emptyset$ for all $a, b \in S$. To distinguish relative $*$-quasi-complementation, we shall write ab for the result of the former, while ae will stay reserved for the latter.

It is clear that $a \in Q_{(a^0,0)}$, so that $(a^0)^0$ can always be chosen in such a way that (1.1.2) is satisfied. If $a \notin Q^*_{(a^0,0)}$ and if it is assumed that the set $\{s : a \leq s, s \in Q_{(a_0,0)}\}$ has a maximum, then that maximum is to be taken for $(a_0)0$; thus, (1.1.2) is again satisfied. Finally, from $Q_{(b,c)} \subseteq Q_{(a,c)}$ and $Q^*_{(b,c)} \subseteq Q^*_{(a,c)}$ when $a \leq b$, it follows that relative quasi-complementation and relative $*$-quasi-complementation can always be defined so that (1.1.3) is satisfied, provided the set $\{s : b_c \leq s, s \in S(a,c)\}$ has a maximum. In that way the following result is proved.

(1.2) Lemma. (i) *Every poset admits at least one relative quasi-complementation.*

(ii) *If S is a poset such that, for all $a, b \in S$, $Q^*_{(a,b)} \neq 0$ and the sets $\{s : a \leq s, s \in Q_{(a_0,0)}\}$, $\{s : b_c \leq s, s \in Q_{(a,c)}\}$ where $a_0 \in Q^*_{(a,0)}$, $b_c \in Q^*_{(b,c)}$, hare maximal elements, then S admits at least one relative $*$-quasi-complementation.*

An element a^b (resp. a_b) is called a quasi (resp. $*$-quasi) complement of a relative to b. A poset S with a relative quasi (resp: $*$-quasi) complementation will be called quasi (resp. $*$-quasi) complemented poset and denoted by S_q (resp. S^*_q). In the sequel we shall write a^{00} instead of $(a^0)^0$.

(1.1) *Definition.* Elements a, b from S_q will be called orthogonal (resp. disjoint) if $a \leq b^0$ (resp. $a \wedge b \leq 0$). The fact that a and b are orthogonal will be written as $a \perp b$.

A poset S_q (resp. S^*_q) will be called complete if it is closed with respect to supremums of all finite collections of its orthogonal elements; a complete S_q (resp. S^*_q) will be denoted by $S_{q,c}$ (resp. $S^*_{q,c}$).

The following statements can be easily verified:

(1.4) *If S is a lattice, $Q^*_{(a,b)}$ has exactly one element for all $a, b \in S$, and there is no element that is uncomparable with that one from $Q^*_{(a,b)}$ and has property* (1.1.1), *then the corresponding $*$-quasi-complemented lattice is a pseudo-Boolean algebra (compare with* [**8**, Ch. IV]*).*

(1.5) *If S_q is such that the inequality in* (1.1.2) *is equality for all $a \in S_q$, then S_q is an orthocomplemented poset (see e. g.* [**1, 4, 5**]*).*

(1.6) *A lattice S^*_q that has all properties from (1.5) and (1.6) is a Boolean algebra.*

A poset such that $Q^*_{(a,b)}$ has exactly one element for all $a, b \in S$ and there is no element with property (1.1.1) that is uncomparable with that one from $Q^*_{(a,b)}$, will be called a pseudo-Boolean poset, and the corresponding relative $*$-quasi-complementation will be called relative pseudo-complementation.

The following result will be needed later.

(1.7) LEMMA. *If $a, b \in S_q$ are such that $a \vee b$ exists, then $a^0 \wedge b^0$ exists and $a^0 \wedge b^0 = (a \vee b)^0$.*

Proof. From $a \leq a \vee b$, $b \leq a \vee b$ and (1.1.3) it follows that $(a \vee b)^0 \leq a^0$, $(a \vee b)^0 \leq b^0$, so that it is enough to prove that, if $c \in S_q$ is such that $c \leq a^0$, $c \leq b^0$, then $c \leq (a \vee b)^0$. But, for each $c \leq a^0$, $c \leq b^0$ because of (1.1.2) and (1.1.3), $a \leq a^{00} \leq c^0$, $b \leq b^{00} \leq c^0$, that is (because of the existence of $a \vee b$) $a \vee b \leq c^0$, which implies $c \leq c^{00} \leq (a \vee b)^0$. The proof is completed.

(1.8) *Definition.* A poset $S_{q,c}$ is called weakly modular if for all $a, b \in S_{q,c}$ such that $a \leq b$, there is a $c \in S_{q,c}$ for which $a \perp c$, $c \leq b$ and $b \wedge (a \vee c)^0 = 0$. It is called modular if that element c is such that $b = a \vee c$ (this property is more often called orthomodularity, see e. g. [**3, 7**]).

Obviously, modularity implies weak modularity, but the converse does not hold. It is easy to see that Boolean algebras are modular, pseudo-Boolean algebras are weakly modular and orthocomplemented posets are in general neither. Also, the assumption that $S_{q,c}$ is modular immediately implies that it is orthocomplemented. Only the following property, somewhat weaker than weak modularity, always holds for $S_{q,c}$.

(1.9) LEMMA. *If $a, b \in S_{q,c}$ are such that $a \leq b$, then there is a $c \in S_{q,c}$ that satisfies conditions $c \perp a$, $c \leq b^{00}$ and $b^{00} \wedge (a \vee c)^0 = 0$. The biggest among such elements c is defined by $c = a^0 \wedge b^{00}$.*

Proof. From $a \leq b \leq b^{00}$ it follows that $a \vee b^0$ exists, which (because of (1.7)) implies the existence of $a^0 \wedge b^{00}$ and the equality $a^0 \wedge b^{00} = (a \vee b^0)^0$. But $a^0 \wedge b^{00} \perp a$, so that $a \vee (a^0 \wedge b^{00})$ exists. From $a^0 \wedge b^{00} \leq b^{00}$ and $a \leq b \leq b^{00}$ one gets $a \vee (a^0 \wedge b^{00}) \leq b^{00}$, which (because $a \vee (a^0 \wedge b^{00}) \leq b^{00}$ means $a \vee (a^0 \wedge b^{00}) \perp b^0$) implies the existence of $(a \vee (a^0 \wedge b^{00}) \vee b^0$, and thus the equality $(a^0 \vee (a^0 \wedge b^{00}))^0 \wedge b^{00} = ((a \vee (a^0 b^{00}) \vee b^0)^0$, which with (1.7) gives $b^{00} \wedge (a \vee (a^0 \wedge b^{00}))^0 = b^{00} \wedge a^0 \wedge (a^0 \wedge b^{00})^0 = 0$, and this proves that the element $c = a^0 \wedge b^{00}$ satisfies the statement. If $d \in S_{q,c}$ is some other element such that $d \perp a$, $d \leq b^{00}$ and $b^{00} \wedge (a \vee d)^0 = 0$, then from $d \perp a$ and $d \leq b^{00}$ one gets, because of the existence of $a^0 \wedge b^{00}$, that $d \leq a^0 \wedge b^{00}$, which completes the proof.

(1.10) COROLLARY. *A complete orthocomplemented poset is weakly modular.*

2. Main result. In this section we shall give a definition of a probability on $S_{q,c}$ and characterize a poset $S_{q,c}$ that admits a set of probabilities which generates the original order in $S_{q,c}$. From that result, in the next section we shall derive some particular results concerning pseudo-Boolean algebras and complete orthocomplemented posets.

(2.1) *Definition.* A function $p : S_{q,c} \to [0, 1]$ will be called a probability on $S_{q,c}$ if the following conditions are satisfied:

(2.1.1) $p(1) = 1$;

(2.1.2) $p(a_1 \vee \cdots \vee a_n) = \sum_{i=1}^{n} p(a_i)$ for each positive integer n and each choice of $a_1, \ldots, a_n \in S_{q,c}$ such that $a_i \perp a_j$, $i \neq j$;

(2.1.3) if $a, b \in S_{q,c}$ are such that $a \leq b$, then $p(a) \leq p(b)$.

Obviously, if $S_{q,c}$ is modular, then (2.1.3) is a redundant property.

Some complete quasi-complemented posets do not admit any probabilities (see e. g. [**3**] and [**1**, Ch. 11]). However, even if probabilities on $S_{q,c}$ do exist, it may happen that the set $\mathcal{P}$ of all probabilities is not rich enough to ensure the following natural and thus desirable property, [**1**]:

(2.2) *If $p(a) \leq p(b)$, for all $p \in \mathcal{P}$, then $a \leq b$.*

A set of probabilities that satisfies this condition will be called full. The following result gives a characterization of a structure $S_{q,c}$ that admits a full set of probabilities (compare with [**7**]).

(2.3) THEOREM. *A poset $S_{q,c}$ admits a full set $\mathcal{P}$ of probabilities if and only if it is isomorphic to a family $\mathcal{F}$ of functions from some set M into $[0,1]$ with the properties:*

(2.3.1) *order in $\mathcal{F}$ is defined pointwise: $f \leq g$ if and only if $f(x) \leq g(x)$ for each $x \in M$;*

(2.3.2) *functions identically equal to zero and one, denoted by 0 and 1, respectively; belong to $\mathcal{F}$;*

(2.3.3) *for all $f, g \in \mathcal{F}$ there is an element in $\mathcal{F}$, denoted by f^g, such that*

(2.3.3.1) *if $h \in \mathcal{F}$ is such that $h \leq f$, $h \leq f^g$, then $h \leq g$;*

(2.3.3.2) *$f \leq f^{00}$ for each $f \in \mathcal{F}$;*

(2.3.3.3) *if $f, g \in \mathcal{F}$ are such that $f \leq g$, then $g^h \leq f^h$ for each $h \in \mathcal{F}$;*

(2.3.4) *if $f_1, f_2, f_3 \in \mathcal{F}$ are such that $f_1 \leq f_2^0$, $f_1 \leq f_3$, $f_2 \leq f_3$, then $f_1 \vee f_2$ exists in $\mathcal{F}$ and is defined by $f_1 \vee f_2 = f_1 + f_2 \leq f_3$,*

Proof. First, let us show that the family $\mathcal{F}$ is a complete quasi-complemented poset with maximal and minimal element. Properties (2.3.1) and (2.3 2) insure that $\mathcal{F}$ is a poset and that it has a maximal and minimal element. A quasi-complementation is obviously defined by (2.3.3) which, together with (2.3.4), implies the existence of $f \wedge f^0$ and $f \vee f^0$ as well as $f \wedge f_0 = 0$ and $f \vee f_0 \leq 1$. Finally, (2.3.4) implies the completeness of $\mathcal{F}$.

Now, let us prove that there is a full set of probabilities on $\mathcal{F}$. For each $x \in M$, let a function p_x from $\mathcal{F}$ into $[0,1]$ be defined by

$$p_x(f) = f(x), \quad f \in \mathcal{F}. \tag{2.4}$$

Set $\mathcal{P} = \{p_x, x \in M\}$ and prove that $\mathcal{P}$ is a full set of probabilities on $\mathcal{F}$. First, from (2.4) and the assumption about functions from $\mathcal{F}$ it follows that $0 \leq p_x \leq 1$ for each $x \in M$. Clearly $p_x(1) = 1$ for each $x \in M$, where 1 on the left hand side is the unit from $\mathcal{F}$ and on the right-hand side just a number. If $f_1, f_2 \in \mathcal{F}$ are orthogonal, then (because of (2.3.4)) $f_1 + f_2 \in \mathcal{F}$, and thus, for an arbitrary $x \in M$, $p_x(f_1 \vee f_2) = (f_1 + f_2)(x) = f_1(x) + f_2(x) = p_x(f_1 + p_x(f_2)$, wich proves (2.1.2). Since $f_1 \leq f_2$ implies $p_x(f_1) = f_1(x) \leq f_2(x) = p_x(f_2)$, the monotony of p_x is also proved. Thus, each function p_x $x \in M$, is a probability on $\mathcal{F}$. It is easy to see that $\mathcal{P}$ determines the order in $\mathcal{F}$. This follows from the fact that the inequality $f_1 \leq f_2$ holds if and only if $f_1(x) \leq f_2(x)$ for each $x \in M$, which can be rewritten as $p_x(f_1) \leq p_x(f_2)$ for each $x \in M$.

To prove the second part of the statement we shall assume that there is a full set of probabilities $\mathcal{P}$ on $S_{q,c}$ and show that there is a set M and a family $\mathcal{F}$ of functions from M into $[0,1]$ which satisfies conditions (2.3.1) – (2.3.4), such that $\mathcal{F}$ and $S_{q,c}$ are isomorphic. Let $\mathcal{F}$ be the set of functions f_a defined in the following way: for each $a \in S_{q,c}$, f_a is a function from $M = \mathcal{P}$ into $[0,1]$ defined by

$$f_a(p) = p(a), \qquad p \in \mathcal{P}. \tag{2.5}$$

Since $\mathcal{P}$ is a full set of probabilities it follows that, if order in $\mathcal{F}$ is defined pointwise, then the mapping $S_{q,c}$ onto $\mathcal{F}$, defined by (2.5), is one-to-one and preserves the order in $\mathcal{F}$. Properties (2.1.1) and (2.1.2) imply the existence of functions of the form (2.5) that are identically equal to zero, that is to one. It is not difficult to see that, if to arbitrary $f_a, f_b \in \mathcal{F}$ one assigns a function $f_a^{f_b}$ defined by $f_a^{f_b} = f_{a^b}$ (in which case we have in particular $f_a^0 = f_{a^0}$), then this mapping has properties (2.3.3.1)–(2.3.3.3). Indeed, if f_c is such that $f_c \leq f_a$, $f_c \leq f_{a^b}$, then the fact that $\mathcal{P}$ is a full set of probabilities implies $f_c \leq f_b$. Also, since $f_a^{00} = f_{a^{00}}$ and $a \leq a^{00}$ is equivalent to $f_a \leq f_{a^{00}}$, property (2.3.3.2) is proved. Property (2.3.3.3) follows from the equivalence of inequalities $f_a \leq f_b$ and $a \leq b$ and from the fact that the latter one implies $b^c \leq a^c$, which in turn is equivalent to $f_b^{f_c} \leq f_a^{f_c}$. Finally, if $f_a, f_b, f_c \in \mathcal{F}$ are such that $f_a \leq f_b^0$, $f_a \leq f_c$, $f_b \leq f_c$, that means that $a \leq b^0$, $a \leq c$, $b \leq c$, which (because of completeness of $S_{q,c}$) implies the existence of $a \vee b$, and thus $a \vee b \leq c$. Hence $f_{a \vee b} \leq f_c$, which (together with (2.5) and (2.1.2)) gives $f_{a\vee b}(p) = p(a \vee b) = p(a) + p(b) = f_a(p) + f_b(p)$, $p \in \mathcal{P}$, so that $f_a \vee f_b$ exists and is defined by $f_a \vee f_b = f_a + f_b \leq f_c$. The proof is completed.

3. Orthocomplemented poset and problems with pseudo-complementation. In this section we shall be concerned about some special forms of the function f^g from (2.3.3) and show how they lead to examples of orthocomplemented and pseudo-Boolean posets. Especially, the main result from [**7**] will be derived as a consequence of Theorem (2.3).

Let us suppose that $f^g \in \mathcal{F}$, satisfying (2.3.3.1), is defined by

$$f^g(x) = \min\{1, 1 - f(x) + g(x)\}, \quad x \in M. \tag{3.1}$$

It is easy to check that it has properties (2.3.3.2) and (2.3.3.3) Namely, since $f^0(x) = 1 - f(x)$, $x \in M$, it is $f^{00} = f$ for each $f \in \mathcal{F}$, so that in fact (3.1) defines an orthocomplemented poset. If $f, g \in \mathcal{F}$ are such that $f \leq g$, then from $1 - g(x) + h(x) \leq 1 - f(x) + h(x)$ it follows that $g^h \leq f^h$ for each $h \in \mathcal{F}$.

(3.2) Lemma. *If the function f^g from* (2.3.3) *is defined by* (3.1)*, then condition* (2.3.4) *is equivalent to*

(3.2.1) *if $f_1, f_2, f_3 \in \mathcal{F}$ are such that $f_i + f_j \leq 1$, $i \neq j$, then $f_1 \vee f_2 \vee f_3$ exists in $\mathcal{F}$ and $f_1 \vee f_2 \vee f_3 = f_1 + f_2 + f_3$.*

Proof. Let us suppose that (2.3.4) holds and that $f_1, f_2, f_3 \in \mathcal{F}$ are such that $f_i + f_j \leq 1$, $i \neq j$. The last assumption, because of (3.1), means that $f_i \leq f_j^0$, $i \neq j$, and thus in particular $f_1 \leq f_2^0$, $f_1 \leq f_3^0$, $f_2 \leq f_3^0$ $i \neq j$, which (together with (2.3.4)) implies the existence of $f_1 \vee f_2$ and $f_1 \vee f_2 = f_1 + f_2 \leq f_3^0$. If (2.3.4) is applied once more, this time to $\overline{f}_1 = f_1 + f_2$, $\overline{f}_2 = f_3$, $\overline{f}_3 = 1$, one gets the existence of $(f_1 + f_2) \vee f_3$ and the equality $(f_1 + f_2)\vee = f_3 = f_1 + f_2 + f_3 \leq 1$. Thus, it is proved that (2.3.4) implies (3.2.1).

Now, let (3.2.1) hold and let f_1, f_2, f_3 be such that $f_1 \leq f_2^0$, $f_1 \leq f_3$, $f_2 \leq f_3$. Since $f_2^0 = 1 - f_2$, the assumption $f_1 \leq f_2^0$ means $f_1 + f_2 \leq 1$. And, because $f_3 = f_3^0$, the inequalities $f_1 \leq f_3$, $f_2 \leq f_3$ become $f_1 + f_3^0 \leq 1$, $f_2 + f_3^0 \leq 1$. If now (3.2.1) is applied, one gets the existence of $f_1 \vee f_2 \vee f_3^0$ and $f_1 \vee f_2 \vee f_3^0 = f_1 + f_2 + f_3^0 \leq 1$, which is equivalent to $f_1 + f_2 \leq 1 - f_3^0$, that is to $f_1 + f_2 \leq f_3^0$, which proves (2.3.4).

(3.3) Lemma. *Let $\mathcal{F}$ be a family of functions from some set M into* $[0,1]$ *satisfying* (2.3.1)–(2.3.4) *and let the function f^g from* (2.3.3) *be defined by* (3.1). *Then $\mathcal{F}$ is a complete modular orthocomplemented poset.*

Proof. Modularity is the only thing which needs proving. Let $f, g \in \mathcal{F}$ he such that $f \leq g$. Then $f \leq g^{00}$ (which (because of (2.3.4) or (3.2.1)) means $f \vee g^0 = f + g^0$ exists, and that with (1.6) and (3.1), implies the existence of $f^0 \wedge g$ and $f^0 \wedge g = (f \vee g^0)^0$. Since $f^0 \wedge g \leq f^0$, $f^0 \wedge g \leq g$, $f \leq g$, it follows (together with (2.3.4) that $(f^0 \wedge g) \vee f$ exists and $(f^0 \wedge g) \vee f = (f^0 \wedge g) + f \leq g$. In Theorem (2.3) it was proved that there is a full set of probabilities on $\mathcal{F}$. If p_x is one of these probabilities)defined by (2.4)), then $p_x((f^0 \wedge g) + f) = p_x(f^0 \wedge g) + p_x(f) = p_x((f + g^0)^0) + p_x(f) = (f + g^0)^0(x) + f(x) = (1 - (f + g^0))(x) + f(x) = 1 - g^0(x) = p_x(g)$. However, since, this equality holds for each p_x and set of probabilties is full, it follows that $(f^0 \wedge g) \vee f = g$, which proves the modularity of $\mathcal{F}$.

(3.4) Corollary ([**7**]). *A complete orthocomplemented poset admits a full set of probabilities if and only if it is isomorphic to a family $\mathcal{F}$ of functions from some set M into* $[0,1]$*, having properties (3.3.1)–(2.3.4), where the function f^g from (2.3.3) is defined by (3.1).*

The next two examples illustrate that not every set of probabilities on a pseudo-Boolean algebra is full (although, as we shall see, a probability on a pseudo-Boolean algebra always exists) and that a characterization of those pseudo-Boolean posets that admit a full set of probabilities cannot be obtained from Theorem (2.3).

(3.5) *Example.* The set $S = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{1,3\}, \{1,2,3\}\}$ becomes a pseudo-Boolean algebra by defining relative pseudo-complementatton in a natural way. A set of probabilities p on S with the property $p(\{1,2\}) = 1$ is not full, because, although $p(\{1,3\}) \leq p(\{1,2\})$, it is not $\{1,3\} \leq \{1,2\}$.

(3.6) *Example.* Let a function $f^g \in \mathcal{F}$, satisfying (2.3.3.1), be defined by

$$f^g(x) = \begin{cases} 1 & \text{if} \quad f(x) \leq g(x) \\ g(x) & \text{if} \quad g(x) \leq f(x). \end{cases} \tag{3.7}$$

It is obvious that (2.3.3.2) is satisfied, as well as (2.3.3.3), because from $f \leq g$ it follows that $f(x) - h(x) \leq g(x) - h(x)$ for each $h \in \mathcal{F}$ and minimal element, then there is an $f \in \mathcal{F}$ such that $f < f^{00}$, $f \neq f^{00}$. It is clear that $f^0 \leq 1 - f$ ($1 - f$ does not have to belong to $\mathcal{F}$). Thus, because $f + f^0 \leq 1$, $f + f^0 \neq 1$, for at least one $f \in \mathcal{F}$ it follows that complementation defined by (3.7) is not an ortho-complementation. It is eas to verify that if for all $f, g \in \mathcal{F}$, f^g is the biggest element from $\mathcal{F}$ which satisfies (2.3.3.1), then $\mathcal{F}$ is a pseudo-Boolean poset. If that is the case, then (2.3.4) is equivalent to the following condition:

(3.8) if $f_1, f_2, f_3 \in \mathcal{F}$ are such that $\{x : f_i(x) > 0\} \cap \{x : f_j(x) > 0\} = \emptyset$ $i \neq j$, then $f_1 \vee f_2 \vee f_3$ exists in in $\mathcal{F}$ and $f_1 \vee f_2 \vee f_3 = f_1 + f_2 + f_3$. It can be seen (without referring to the general theory of pseudo-Boolean algebras) that, if $\mathcal{F}$ has a lattice structure and pseudo-complementation is defined by (3.7), then it is weakly modular, although not modular.

Since the function f^g, defined by (3.7), is not the only one that can serve as pseudo-complementation it is not clear how one should formulate a theorem, corresponding to (2.3), that would characterize those pseudo-Boolean posets which admit full sets of probabilities. In the case of pseudo-Boolean algebras this problem turns out to be easy. Namely, since it is known that each pseudo-Boolean algebra is isomorphic to a family of all open sets of some set-theoretical topological Boolean algebra [**8**, Ch. IV] and since every Boolean algebra admits a full set of probabilities, it follows that every pseudo-Boolean algebra admits such a set, too [**2**]. Consequently, the following result is proved.

(3.9) Lemma. *Every pseudo-Boolean algebra is isomorphic to a lattice of functions that satisfy conditions* (2.3.1)–(2.3.3) *and in which the supremum of orthogonal functions is defined as a summation.*

However, that does not solve the problem of the existence of a full set of probabilities on a pseudo-Boolean poset (which is not an algebra). That problem will de discussed in the next section.

4. Pseudo-complemented posets. In this section sufficient conditions for the existence of a full set of probabilities on a pseudo-Boolean poset will be given.

Let S be a pseudo-Bollean poset that satisfies the following conditions:

(4.1) for all arbitrary $a, b \in S$ such that $a \leq b$, $a \neq 0$, $b \neq 1$, the set $\{c \in S : a \leq c \leq b\}$ is finite;

(4.2) all antichains are finite.

Let $a_1, \ldots, a_n \in S$ be elements whose infimum does not exist. Then (4.2) implies that there are elements $c_1, \ldots, c_m \in S$ with the property:

(4.3) $c_i < a_j$, $c_i \neq a_j$ for all $j = 1, \ldots, n$ and $i = 1, \ldots, m$, and there is no $d \in S$ such that $c_i < a$, $c_i \neq d \leq a_j$ for some i and all $j = 1, \ldots, n$. For each $i = 1, \ldots, m$ there are elements $b_1^i, \ldots, b_{\lambda_i}^i \in S$ such that:

(4.4) $c_i < b_j^i$ $c_i \neq b_j^i$ for $j = 1, \ldots, \lambda_i$;

(4.5) for each $j = 1, \ldots, \lambda_i$, there is at least one $k = k(j)$ such that $b_j^i \leq a_k$;

(4.6) for each $i = 1, \dots, m$, there is at least one $l = l(i)$ such that $b_i^i \leq a_i$;

(4.7) $b^i \wedge \cdots \wedge b_{\lambda_i}^i = c_i$;

(4.8) for each $j = 1, \dots, \lambda_i$, if d is such that $c_i \leq d \leq b_j^i$, then either $d = c_i$ or $d = b_j^i$.

It might happen that some or all among the elements $b_1^i, \dots, b_{\lambda_i}^i$, are equal to $a_1, \dots, a_n$, respectively (if the latter is the case, then clearly $\lambda = n$).

Indeed, since (because of (4.1)) there are only finitely many chains connecting a_j and c_i for any fixed i and j, then the number of chains whose minimal element is c_i and whose maximal element is from the set $\{a_1, \dots, a_n\}$ is also finite. From (4.1) it also follows that in each of these chains there is a first successor of c_i (some of these first successors may be equal to each other, and some of them may coincide with one of the elements $a_1, \dots, a_n$). If different elements among those first successors are denoted by $b_1^i, \dots, b_{\lambda_i}^i$, then it is immediately clear that they have properties (4.4)–(4.8).

Similary, if $a_1, \dots, a_n \in S$ are elements whose supremum does not exist, and if $d_1, \dots, d_m \in S$ are elements satisfying:

(4.9) $a_j < d_i$, $a_j \neq d_i$ for all $j = 1, \dots, n$ and $i = 1, \dots, m$ and there is no $e \in S$ such that $a_j \leq e < d_i$, $e \neq d_i$ for some i and all $j = 1, \dots, n$, then for each i there are elements $e_1^i, \dots, e_{\lambda_{i'}}^i$ such that:

(4.10) $e_j^i < d_i$, $e_j^i \neq d_i$ for each $j = 1, \dots, \lambda_{1'}$;

(4.11) for each $j = 1, \dots, \lambda_{i'}$, there is at least one $k = k(j)$ such that $a_k \leq e_j^i$;

(4.12) for each $i = 1, \dots, n$, there is at least one $l = l(i)$ such that $a_i \leq e_1^i$;

(4.13) $e_1^i \vee \cdots \vee e_{\lambda_{i'}}^i = d_i$;

(4.14) for each $j = 1, \dots, \lambda_i'$, if f is such that $e_j^i \leq f \leq d_i$, then either $f = d_i$ or $f = e_j^i$.

It might happen that some or all among the elements $e_1^i, \dots, e_{\lambda_i'}^i$ are equal to $a_1, \dots, a_n$, respectively (in the latter case $\lambda_i = n$).

Let S be a poset with pseudo-complementation and let for $a, b \in S$, the pseudo-complement of a relative to b be denoted by a^b. We shall show that S is isomorphic to a subset of a pseudo-Boolean algebra S_p with the property that, if $a, b \in S$, then the pseudo-complement of a relative to b in S_p is equal to a^b.

To achieve this we shall enlarge a poset S in the following way. If $a_1, \dots, a_n$ are elements from S satisfying conditions:

(4.15) the infinium of any proper subset of elements from $\{a_1, \dots a_n\}$ does not exist in S;

(4.16) $a_1 \wedge \dots a_n$ does not exist either in S or in the set of previously added elements, then an element e such that

(4.17) $c_i \leq e \leq b_j^i$ for all $i = 1, \dots, m$ and $j = 1, \dots, \lambda_i$,

is added to S.

Similarly, if $a_1, \ldots, a_n$ are elements from S with the properties

(4.18) the supremum of any subset of elements from $\{a_1, \ldots, a_n\}$ does not exist in S;

(4.19) $a_1 \vee \cdots \vee a_n$ does not exist either in S or in the set of previously added elements,

then an element f such that

(4.20) $e^i_j \leq f \leq d_i$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, \lambda'_i$,

is added to S.

It is obvious that this enlargement procedure is not unique.

(4.21) LEMMA. *The set S', consisting of all elements from S and all elements added in the above way, is a lattice.*

Proof. For simplicity, only proofs for supremums and infimums of two elements will be derived. Proofs for general cases will then be obvious. If a and b both belong to $S' \setminus S$, then there are elements $a_1, \ldots, a_n, a'_1, \ldots, a'_n$, and $b_1, \ldots, b_m, b'_1, \ldots b'_m$ from S such that:

(4.22) $a_i < a < a'_j$ $a_i \neq a \neq a'_j$ and $b_i < b < b'_j$ $b_i \neq b \neq b'_j$ for all i, j;

(4.23) a (resp. b) is the smallest proper successor of $a_1, \ldots, a_n$ (resp. $b_1, \ldots, b_m$) and the biggest proper predecessor of $a'_1, \ldots a'_n$, (resp. $b'_1, \ldots, b'_m$);

(4.24) there is no other element in S whose smallest proper successor is a (resp. b) or which has a (resp. b) for its biggest proper predeccessor

If the infimum of the elements $a_1, \ldots, a_n, b_1, \ldots, b_m$ exists in S', then the infimum of a and b also exists and these two are equal. However, if $a_1 \wedge \cdots \wedge a_n \wedge b_1 \wedge \cdots \wedge b_m$ does not exist in S has been already enlarged by that infimum. Thus $a \wedge b$ exists. An analogous argument applies for $a \vee b$, except that this time the elements $a'_1, \ldots, a'_{n'}, b'_1, \ldots, b'_{m'}$ are used.

Finally, if, for example, $a \in S$ and $b \in S' \setminus S$, the previous argument should be used for $a, b_1, \ldots, b_m$, that is $a, b'_1, \ldots, b'_m$

The proof is completed.

Now we can give sufficient conditions for the enlarged set S' not to alter pseudo-complements of elements from S (obtained with respect to S).

(4.25) THEOREM. *If a lattice S' is distributive, then it is a pseudo-Boolean algebra in which the following holds:*

(4.25.1) *If a and b are from S, then a pseudo-complement of a relative to b in S' coincides with a^b.*

Proof. First we shall prove that (4.25.1) holds. It is obvious that $a \wedge a^b \leq b$, so that we must prove that if there is a $c \in S'$ such that $a \wedge c \leq b$, then $c \leq a^b$. If $c \in S'$ such that $a \wedge c \leq b$ exists, then it is obvious that we must have $c \in S' \setminus S$, and that means that there are elements $c_1, \ldots, c_n \in S$ such that

(4.26) $c_i < c$, $c_i \neq c$ for each $i = 1, \ldots, n$;

(4.27) c is the smallest proper successor of $c_1, \dots, c_n$;

(4.28) there is no other element whose smallest proper successor is a.

Since $a \wedge c \leq b$ and $a \wedge c_i \leq a \wedge c$, it follows that $a \wedge c_i \leq b$ for each $i = 1, \dots, n$. However, since a^b is unique in S, it follows that $c_i \leq a^b$ for each $i = 1, \dots, n$, whach implies that either $c < a^b$, $c \neq a^b$ (which is what we wanted to prove) or $c_i = a^b$ for each $i = 1, \dots, n$, which contradicts the assumption that $c \in S' \setminus S$. Thus, (4.25.1) is proved.

For the proof of the remaining part we shall need the following two simple statements that hold on any distributive lattice L;

(4.29) If $a, b \in L$ are such that d^a and d^b exist and if $a \wedge b = c$, then d^c exists and $d^c = d^a \wedge d^b$;

(4.30) If $a, b \in L$ are such that a^b and b^d exists and if $a \vee b = c$, then c^d exists and $c^d = a^d \wedge b^d$.

Let us first prove (4.29). If $d(c) \in L$ is any element such that $d \wedge d(c) \leq c$, then from $c \leq a$, $c \leq b$ it follows that $d(c) \leq d^a$, $d(c) \leq d^b$, so that $d(c) \leq d^a \wedge d^b$. Let $e = d^a \wedge d^b$. From $e \leq d^a$, $e \leq d^b$ it follows that $e \wedge d \leq d^a \wedge d \leq a$, $e \wedge d \leq d^b \wedge d \leq b$, so that $e \wedge d \leq a \wedge b = c$, which (together with the inequality $d(c) \leq d^a \wedge d^b$) implies the existence of d^c and $d^c = d^a \wedge d^b$. Thus (4.29) is proved.

To prove (4.30) we shall suppose that $c(d)$ is any element such that $c \wedge c(d) \leq d$. Then from $a \leq c$, $b \leq c$ it follows that $c(d) \leq a^d$, $c(d) \leq b^d$, so that $c(d) \leq a^d \wedge b^d$. From the distributivity of L one gets $(a^d \wedge b^d) \wedge (a \vee b) = (a^d \wedge b^d \wedge a) \vee (a^d \wedge b^d \wedge b) \leq (a^d \wedge a) \vee (b^d \wedge b) \leq d$, which (together with the inequality $c(d) \leq a^d \wedge b^d$) implies the existence of c^d and $c^d = a^d \wedge b^d$. Thus (4.30) is proved.

Now, let us suppose that $a \in S$, $b \in S' \setminus S$ and let us prove that a^b exists in S'. Since $b = b'_1 \wedge \dots \wedge b'_{m'}$, where the elements $b'_1, \dots, b'_{m'} \in S$ are from (4.22)–(4.24), and since $a^{b_{j'}}$ exists for each $j = 1, \dots, m'$, statement (4.29) implies the existence of a^b and the equality $a^b = a^{b'_1} \wedge \dots \wedge a^{b'_{m'}}$.

It is easy to prove that b^a exists. Namely, since $b = b_1 \vee \dots \vee b_m$, where $b_1, \dots, b_m$ are from (4.22)–(4.24), and since b_j^a exists for each $j = 1, \dots, m$, statement (4.30) immediately immediately implies the existence of b^a and $b^a = b_j^a \wedge \dots \wedge b_m^a$.

Finally, let a and b be both from $S' \setminus S$. Then (4.30) implies that $a_j^{b'}$ exists and $a_j^{b'} = a_1 b'_j \wedge \dots \wedge a_n^{b'_j}$, $j = 1, \dots, m'$. (because $a_1 \vee \dots \vee a_n = a$ and $a_i^{b'_j}$ exists for all $i = 1, \dots, n$ and $j = 1, \dots, m'$). Now from (4.25.1), (4.29) and $b = b'_1 \wedge \dots \wedge b'_{m'}$ one gets the existence of a^b and

$$a^b = a^{b'_1 \wedge \dots \wedge b'_{m'}} = ab^{b'_1} \wedge \dots \wedge a^{b'_{m'}}.$$

The proof is completed.

This result and the fact that a pseudo-Boolean algebra admits a full set of probabilities (see the previous section and [**2**]) immediately imply the following theorem giving sufficient conditions for a pseudo-Boolean poset to admit a full set of probabilities.

(4.31) THEOREM. *Let a pseudo-Boolen poset satisfy conditions* (4.1) *and* (4.2). *If a lattice S', obtained by an enlargement of S according to* (4.15)–(4.18), *is distributive, then S admits a full set of probabilities.*

(4.32) *Example.* It is not difficult to check that a family $\{0, a_1, a_2, a_3, a_4, 1\}$ in which the only relationships between the elements (except obvious ones involving 0 or 1) are $a_1 \leq a_3$, $a_1 \leq a_4$, $a_2 \leq a_3$, $a_2 \leq a_4$, is a pseudo-Boolean poset. However it is not a lattice, because $a_1 \vee a_2$ and $a_3 \wedge a_4$ do not exist. A natural enlargement gives a pseudo-Boolean algebra which is isometric to the family of all open elements of a topological Boolean algebra consisting of all subsets of a set $\{1, 2, 3, 4\}$, in which an interior operation I is defined by: $I(\emptyset) = I(\{3\}) = I(\{4\}) = I(\{3, 4\}) = \emptyset$, $I(\{1\}) = I(\{1, 3\}) = I(\{1, 4\}) = I(\{1, 3, 4\}) = \{1\}$, $I(\{2\}) = I(\{2, 3\}) = I(\{2, 4\}) = I(\{2, 3, 4\}) = \{2\}$, $I(\{1, 2\}) = \{1, 2\}$, $I(\{1, 2, 3\}) = \{1, 2, 3\}$, $I(\{1, 2, 3, 4\}) = \{1, 2, 3, 4\}$, $I(\{1, 2, 4\}) = \{1, 2, 4\}$.

The question whether a pseudo-Boolean poset always admits a distributive enlargement (of the described type) is still open. While it seems that condition (4.1) is essential for the described enlargement procedure to be meaningful, it is likely that condition (4.2) can be weakened or perhaps completely omitted.

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(Received 19 03 1985)