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EXTENDING DERIVATIONS AND ENDOMORPHISMS TO SKEW POLYNOMIAL RINGS

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Abstract. We treat the problem of extending derivations and endomorphisms of a given ring R to a skew polynomial ring $R[x, f, d]$ over R. As an application we obtain the general conditions for the existence of such rings in finitely many variables over R. We also prove that under suitable conditions, the d (or the f)-simplicity of R implies the f-simplicity of $R[x, f, d]$.

In [**3**] we have obtained the general conditions for the existence of skew polynomial rings of the form $R[x_1, d_1] \dots [x_n, d_n]$ over R and we have treated the problem of the simplicity of such rings.

1. Preliminaries. All the rings considered in this paper are with identities. Let R be a ring and let f be an endomorphism of R. Then a map $d : R \to R$, such that $d(a + b) = d(a) + d(b)$ and $d(ab) = ad(b) + d(a)f(b)$ for all a, b in R, is called a f-derivation of R; when f is the identity map of R, then d is a derivation of R (inner or outer, cf. [**3**, Section 1]).

Given an automorphism f of R, an ideal I is said to be a f-ideal of R if $f(I) = I$ and R is said to be a f-simple ring if it has no non-zero proper f-ideals. The notion of a d-simple ring R (where d is a derivation of R) is defined in a similar way (cf. [**3**, Section 1]).

Assume next that f is an endomorphism and that d is a f-derivation of R. Consider the set S of all polynomials in one variable, say x, over R and define in S addition in the usual way and multiplication by the rule $xr = f(r)x + d(r)$, for all r in R. Then it is well known that S becomes a ring denoted by $R[x, f, d]$ and called a skew polynomial ring or an Ore extension over R (e.g. [**1**, p. 35]).

Applying induction on n one finds that $x_n r = \sum_{i=0}^{n} \binom{n}{i} f^{n-i}(d^i(r)) x^{n-i}$, for all r in R. When f is the identity map of R we write $S = R[x, d]$ and when d is the zero derivation of R we write $S = R[x, f]$.

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2. Main results. First we need the following lemma:

LEMMA 2.1. *Let R be a ring and let $S = R[x, f, d]$ be a skew polynomial ring over R. Assume that f is a monomorphism of R. Then x is a regular element of S.*

Proof. It suffices to show that if r is in R and $xr = 0$, then $r = 0$. Assume the contrary, then $f(r)x = -d(r)$ is in R, therefore $f(r) = 0$, which is absurd.

We are now ready to prove

THEOREM 2.2. *Let R and S be as in the previous Lemma. Then an endomorphism g of R extends to an endomorphism of S by $g(x) = x$ if, and only if, g commutes with f and d.*

Proof. Assume that g extends as above. Then for all r in R $g(xr) = g(f(r)x) + g(d(r))$, or $xg(r) = g(f(r))x + g(d(r))$. But $xg(r) = f(g(r))x + d(g(r))$, therefore $[g(f(r)) - f(g(r))]x = d(g(r)) - g(d(r))$ is in R. Hence, by Lemma 2.1, $f \circ g = g \circ f$ and $d \circ g = g \circ d$. Conversely assume that g commutes with f and d. Then g extends to an endomorphism of S if $g(x)$ can be defined in a way compatible with the multiplication in S. But the previous relations give that $g(xr) - xg(r) = g(f(r)x) - f(g(r))x$, therefore we can put $g(x) = x$.

THEOREM 2.3. *Let R, S and g be as in the previous statement and let d' be a g-derivation of R. Then d' extends to a g-derivation of S by $d'(x) = 0$ if and only if, d' commutes with f and d.*

Proof. By the previous theorem g extends to an endomorphism of S by $g(x) = x$. Assume first that d' extends as above. Then for all r in R $d'(xr) = d'(f(r)x) + d'(d(r))$ (1), or $xd'(r) = d'(f(r)x + d'(d(r))$. But $xd'(r) = f(d'(r))x + d(d'(r))$ (2), therefore $[f(d'(r)) - d'(f(r))]x = d'(d(r)) - d(d'(r))$ is in R. Thus $f \circ d' = d' \circ f$ and $d \circ d' = d' \circ d$. Conversely assume that d' commutes with d and f, then d' extends to a g-derivation of S if $d'(x)$ can be defined so that it satisfies relation (1). Namely, if $d'(x) = h$, we should be able to write $xd'(r) + hg(r) - f(r)h + d'(f(r))x + d'(d(r))$. Then relation (2) gives that $hg(r) = f(r)h$, hence we can put $d'(x) = 0$.

Our next result generalizes Theorem 2.2 of [**3**].

THEOREM. 2.4. *Let R be a ring, let $f_1, \ldots, f_n$ be monomorphisms of R and let d_1 be a f_i-derivation of R, for each $i = 1, \ldots, n$. Consider the set S_n of all polynomials in n variables, say $x_1, \ldots, x_n$, over R. Define in S_n addition in the usual way and define multiplication by the relations $x_i r = f_i(r)x_i + d_i(r)$ and $x_i x_j = x_j x_i$ for all r in R and all $i, j = 1, \ldots, n$. Then S, is an Ore extension over S_{i-1} (where $S_0 = R$) if, and only if, $d_i \circ d_j = d_j \circ d_i$, $f_i \circ f_j = f_j \circ f_i$ and $f_i \circ d_j = d_j \circ f_i$, for all $i, j = 1, \ldots, n$.*

Proof. Use Theorems 2.2 and 2.3 and apply induction on n.

We call the ring constructed above a skew polynomial ring in finitely many variables over R and we denote it by $S_n = R[x_1, f_1, d_1] \ldots [x_n, f_n, d_n]$.

Our last Theorem gives a method of constructing f simple skew polynomial rings over R.

THEOREM 2.5. *Let R be an integral domain of characteristic zero, let f be an automorphism of R and let d be a f derivation of R commuting with f. Assume that there exists a central element r_0 of R such that $f(r_0) = r_0$ and $d(r_0) \neq 0$. Then, if R is either a d-simple or a f-simple ring, $S = R[x, f, d]$ is a f-simple ring.*

Proof. By Theorem 2.2 f extends to an automorphism of S by $f(x) = x$. Assume first that R is a d-simple ring and let I be a non zero f-ideal of S. Then, for all r in $I \cap R$, $d(r) = xr - f(r)x$ is in $I \cap R$, therefore $I \cap R$ is a d-ideal of R.

Let $g = \sum_{i=0}^{n} a_i x^i$ be element of I, then $r_0 g - g f^{-n}(r_0) = r_0 a_{n-1} x^{n-1} - [n a_n f^{-1}(d(r_0)) + a_{n-1} f^{-1}(r_0)] x^{n-1} +$ terms of lower degree $= -n a_n d(r_0) x^{n-1} +$ terms of lower degree, and therefore I contains an element of degree $n-1$. Repeating the same argument we finally get that $I \cap R \neq \{0\}$, therefore $I \cap R = R$ and $I = S$. Assume next that R is a f-simple ring, then obviously $I \cap R$ is a f-ideal of R. Show as before that $I \cap R \neq \{0\}$ to get the required result.

3. Remarks and Examples. 1) Let $R = T[y]$ be a polynomial ring over a given ring T. Put $d/dy = d$ and consider the first Weyl Algebra $A_1(T) = R[x, d]$ over T. Since y is a central element of R and $dy = 1$, d is an outer derivation of R. but its extension to $A_1(T)$ by $d(x) = 0$ is an inner derivation induced by x.

2) In general, if d is a derivation of a given ring R and d' is another derivation of R extending to an inner derivation of $S = R[x, d]$ induced by an element $h = \sum_{i=0}^{n} a_i x^i$ of S, then a_n is a central element of R and $r a_k - a_k r = \sum_{i=k+1}^{n} a_i \binom{i}{i-k} d^{i-k}(r)$, for all r in R and each $k = 0, 1, \ldots, n-1$. This is a straightforward consequence of the fact that $d'(r) = hr - rh$ is in R, for all r in R.

3) Let $S = R[x, f]$ be a skew polynomial ring defined with respect to an endomorphism f of a given ring R.

Assume that there exists an outer derivation d of R extending to an inner derivation of S induced by $h = \sum_{i=0}^{n} a_i x^i$. Then, for all r in R, $d(r) = \sum_{i=1}^{n} [a_i f^i(r) - r a_i] x^i + (a_0 r - r a_0)$ is in R, therefore $d(r) = a_0 r - r a_0$, which contradicts our hypothesis.

4) Let g be an endomorphism and let d be a g-derivation of a given ring T. Then g extends to an endomorphism of $A_1(T)$ by $g(x) = x$ and d extends to a g-derivation of $A_1(T)$ by $d(x) = 0$. For this, it is clear that g extends to an endomrphism of $T[y]$ by $g(y) = y$ and that d extends to a g-derivation of $T[y]$ by $d(y) = 0$. It is easy also to check that d/dy commutes with g and d, therefore the result follows by Theorems 2.2 and 2.3.

5) Let $R = T[y_1, \ldots, y_n]$ be a polynomial ring over 2 given ring T. Define the T-automorphism f_1 of R by the relations $f_i(y_j) = y_j + 1$ if $i \neq j$ and $f_i(y_i) = y_i$

for each $i = 1, \dots, n$, and let d_i be the f-derivation of R defined by the relations $d_i(t) = 0$ for all t in T, $d_i(y_j) = 0$ if $i \neq j$ and $d_i(y_i) = 1$. It is easy to check that $d_i \circ d_j = d_j \circ d_i$, $f_i \circ f_j = f_j \circ f_i$ and $f_i \circ d_j = d_j \circ f_i$ for all $i, j = 1, \dots, n$, therefore we can construct the skew polynomial ring $S_n = R[x_1, f_1, d_1] \dots [x_n, f_n, d_n]$ (cf. Theorem 2.4). If we replace f_i with the identity map of R for each i, then S_n is the n-th Weyl Algebra $A_n(T)$ over T.

6) Notice that if f is the identity map of R, Theorem 2.5 gives that if R is d-simple then S is simple, but this is a straightforward consequence of Theorem 3.4 of [**3**].

7) Let $k(y)$ be the field of rational functions over a field k of characteristic zero and let $R = k(y)[t]$ be a polynomial ring over $k(y)$. Define a $k(y)$-automorphism of R by $f(t) = t + 1$. Then R is an f-simple ring (cf. [**2**, Theorem 2.1.1]).

Extend the derivation $d = d/dy$ of $k(y)$ to a f-derivation of R by $d(t) = 0$. Each element of R can be written in the form g/b with g in $k[y][t]$ and b in $k[y]$. Then it is easy to check that $f(d(g/b)) = g^*(t+1)b - g(t+1)b'/b^2$, where g^* denotes the polynomial obtained from g by replacing its coefficients (which are polynomials in y) with their usual derivatives and b' denotes the derivative of b. Hence $d(f(g/b)) = d(g(t+1)/b) = f(d(g/b))$, therefore f commutes with d. Furthermore $f(y) = y$, $d(y) = 1$ and R is a commutative integral domain, therefore, by Theorem 2.5, $R[x, f, d]$ is a f-simple ring.

8) Let $k(y)$ and R be as in the previous example. Define a k-automorphism of $k(y)$ by $f(y) = y + 1$ and extend f to an automorphism of R by $f(t) = t$. Let d be the f derivation of R defined by the relations $d(c) = 0$ for all c in k, $d(y) = 0$ and $d(t) = 1$. Then it is easy to check that R is a d-simple ring and that f commutes with d. Therefore, since $f(t) = t$ and $d(t) \neq 0$, Theorem 2.5 gives that $R[x, f, d]$ is f-simple ring.

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