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AN APPLICATION OF CIRCUIT POLYNOMIALS TO THE COUNTING OF SPANNING TREES IN GRAPHS

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Abstract. t is shown that the number of spanning trees in a graph can be obtained from the circuit polynomial of an associated graph. From this, the number of spanning tress in a regular graph is shown to be obtainable from the characteristic polynomial of a node-deleted subgraph. Finally, Cayley's theorem for the number of labelled tress is derived.

1. Introduction

The graphs considered here will be finite and may contain loops and multiple edges. Let G be a graph. A *circuit* or *cycle corer* in G, is a spanning subgraph of G whose components are all cycles. We will take an isolated node, or in some cases, a loop, to be a cycle with one node. A cycle with two nodes will either be an edge, or in some cases, the multigraph consisting of two nodes joined by a pair of edges. Cycles with more than two nodes will be called *proper cycles*. A cycle with r nodes will be denoted by Z_r.

Let us associate an indeterminate or *weight* w_α with every cycle a in G, and with every cycle cover S, the weight $w(S) = \prod_{w_\alpha}$, where the product is taken over all the components of S. Then the *circuit polynomial* of G is $C(G;\mathbf{w}) = \sum w(S)$, where $\mathbf{w}$ is a vector of weights and the summation is taken over all the cycle covers in G. The basic properties of circuit polynomials are given in Farrell [**4**].

In this paper, we will obtain the number of spanning tress in a graph as a specially weighted circuit polynomial. From this, we will deduce that the number of spanning trees in a regular graph can be obtained from the characteristic polynomial of an associated graph. We will then derive Cayley's formula for the number of labelled trees with p nodes.

Let G be a graph and v_i node of G. By $G - v_i$, or simply G' (when it is unnecessary to specify v_i), we will mean, the graph obtained from G by removing node v_i. The characteristic polynomial of G will be denoted by $\varphi(G;x)$ where x is an indeterminate. The number of spanning trees in G will be denoted by $\tau(G)$.

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2. Preliminary Results

In this section, we will give, without proofs, some results which will be vital to the material which follows. The relation between the circuit polynomial and the characteristic polynomial of a graph was established in Farrell [**3**]. It is given in the following lemma.

Lemma 1. *Let G be a graph and $C(G;\mathbf{w})$ its circuit polynomial in which a weight w_r is given to each circuit with r nodes. Then*

$$\varphi(G;x) = C(G;(x,-1,-2,-2,-2,\ldots,-2)).$$

The following result was derived in Farell and Grell [**5**]. It establishes a connection between circuit polynomials and determinants of matrices.

Lemma 2. *$M = (m_{ij})$ be a $p \times p$ symmetric matrix. Let G_m be a graph associated with M as follows. G_m has node set $\{v_1, v_2, \ldots, v_n\}$. Nodes v_i and v_j are joined by an edge labelled m_{ij}, iff $m_{ij} \neq 0$. Also, at each node v_i of G_m there is a loop labelled m_{ii}. Then $|M| = C(G_m;\mathbf{w})$, where $w_{v_i} = m_{ii}$, $w_{v_i v_j} = -m_{ij}^2$ and $w(Z_r) = (-1)^{r+1} 2 \prod_{k=1}^{r} m_{i_k j_k}$, where $m_{i_k j_k}$ $(k = 1, 2, \ldots, r)$ are the labels on the edges of Z_r.*

This lemma has been further generalized to cover all square matrices. A full discussion of the lemma and its generalizations can be found in [**5**]. We note that when applying this lemma, a circuit with one node is taken to be a loop.

3. The Main Theorems

Let G be a labelled graph with p nodes and with adjacency matrix A. Let $M = (m_{ij})$ be the $p \times p$ matrix obtained from $-A$, by replacing a_{ii} by d_i, the degree of node i. The famous Matrix-Tree theorem (due to Kirchhoff [**8**]) states that all the $(p-1) \times (p-1)$ principal minors of M are numerically equal, and that their common value is the number of spanning trees in G.

Since M is symmetric, its determinant can be evaluated by using Lemma 2. All we need to do now, is to find a relation between the given graph G and the graph G_m. Suppose that G has n_{ij} edges joining nodes i and j. Then we will have $m_{ij} = -n_{ij}$; and in G_m nodes v_i and v_j will be joined by, an edge, labelled $-n_{ij}$. Since $m_{ii} = d_i$, it follows that each node v_i of G_m, will have a loop labelled d_i. Hence we have the following lemma.

Lemma 3. *Let G be a labelled graph with p nodes and possibly with multiple edges. Let M be the matrix associated with G by the Matrix-Tree theorem. Then the graph G_m associated with $|M|$ by Lemma 2, is obtained, from G as follows:*

(i) *Replace the n_{ij} edges joinig nodes i and j by a single edge $v_i v_j$ labelled $-n_{ij}$.*

(ii) *Add a loop labelled d_i to each node i of G.*

This lemma, together with Lemma 2, yield the following theorem.

THEOREM 1. *Let G be a graph, possibly with multiple edges. Let n_{ij} be the number of edges joining nodes i and j. Let G_m be the graph obtained from G as described in Lemma 3. Then $\tau(G) = C(G_m - v; \mathbf{w})$, for any node v of G_m, where $w_{v_i} = di$, $w_{v_i v v_j} = -n_{ij}^2$ and $w(Z_r) = -2\prod_{k=1}^{r} n_{i_k j_k}$.*

Proof. It can be easily seen that the removal of node v_i from G_m yields a graph G'_m associated with a first principal minor of M, since this operation does not affect any of the labels in the subgraph G'_m. Therefore the result follows from Lemma 2. □

We will call two graphs cocircuit, if they have the same circuit polynomial (with respect to a given fixed weight). The following corollary is immediate from the Matrix-Tree theorem.

COROLLARY 1.1. *The node-deleted subgraphs of G_m are all cocircuit graphs under the weight assignment defined in the theorem.*

This corollary suggests that the graph G'_m can be judiciously chosen so as to minimize the amount of computations involved in applying Theorem 1.

Suppose that G is a *strict graph* i.e. G has no loops nor multiple edges. Then G_m is obtained from G by putting labels of -1 on each edge and attaching a loop labelled d_i to each node i of G. Since G_m contains no multiple edges, we can assign weights uniquely as follows: A cycle with $r\,(>1)$ nodes will be given a weight w_r, i.e. $w(Z_r) = w_r$ for $r > 1$. In this case we again take Z_1 to be an isolated loop and asign the weight d_i – the degree of node i in G, to loop v_i in G_m. Our discussion leads to the following theorem.

THEOREM 2. *Let G be a strict graph, and let $\mathbf{w} = (w_{v_i}, w_2, w_3, \dots)$. Then*

$$\tau(G) = C(G'_m;\, (d_i, -1, -2, -2, \dots, -2))$$

i.e. the circuit polynomial in which node v_i is given the weight d_i, edges are given the weight -1 and proper cycles are given the weight -2.

Proof. The result follows immediately from Theorem 1, by putting $n_{ij} = 1$, for $i \neq j$ and i, j adjacent in G. □

In the case of a strict d-regular graph, it will be unnecessary to add loops to G. Instead, each node can be given a weight d, and G' used instead of G'_m.

COROLLARY 2.1. *Let G be a strict graph, regular of degree d. Then*

$$\tau(G) = C(G';\, (d, -1, -2, -2, \dots, -2)).$$

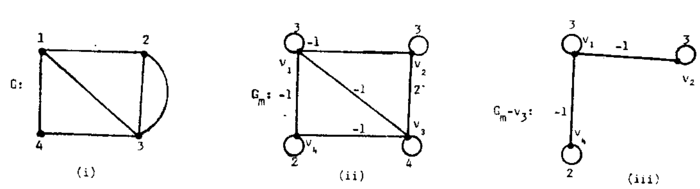

Figure 1.

The cycle covers of $G_m - v_3$ are (i) the three loops; (ii) edge v_1v_2 and the loop at v_4; (iii) edge v_1v_4 and the loop at v_2. The weights of these covers are (i) $3 \cdot 3 \cdot 2 = 18$; (ii) $-1 \cdot 2 = -2$ (iii) $-1 \cdot 3 = -3$. Hence we get from Theorem 1: $\tau(G) = C(G'_m; \mathbf{w}) = 18 - 2 - 3 = 13$.

Notice that $G_m - v_4$ is the triangle with loops at each of its nodes. It can be easily verified that

$$C(G_m - v_4; \mathbf{w}) = 3 \cdot 3 \cdot 4 - (2^2 \cdot 3 + 1^2 \cdot 3 + 1^2 \cdot 4) + 2(-2)(-1)(-1) = 13,$$

in agreement with Corollary 1.1.

4. Some Applications

The following theorem is immediate from Lemma 1 and Corollary 2.1.

THEOREM 3. *Let G be a strict graph, regular of degree x. Then $\tau(G) = \varphi(G; x)$.*

Theorem 3 is a useful result in Spectral Theory. It was first mentioned by Hutschenreuter [**7**], and was used by Cvetković [**1**] to determine the number of spanning trees in several classes of regular graphs. This result has since been extended to non-regular graphs by Cvetković and Gutman [**2**].

Let us denote by K_p, the complete graph with p nodes. The characteristic polynomial of K_p is given by Harary et al [**6**] as

$$\varphi(K_p; x) - (1 + x - p)(1 + x)^{p-1} \tag{1}$$

An independent derivation of this result from the corresponding result for circuit polynomials is given in [**4**]. We can use Equation (1) to derive Cayley's famous formula, for the number of labelled trees with p nodes.

THEOREM 4. *The number of labelled trees on p nodes is p^{p-2}.*

Proof. Each labelled tree on p nodes is a spanning tree of K_p, and vice versa. Therefore the number of labelled trees on p nodes is $\tau(K_p)$. But K_p is strict and regular of degree $p - 1$. Therefore from Theorem 3, we get

$$\tau(K_p) = \varphi(K_{p-1}; p - 1) = (1 + (p - 1) - (p - 1))p^{p-2} = p^{p-2}. \quad \square$$

REFERENCES

[1] D. M. Cvetković, *The spectral method for determining the number of trees*, Publ. Inst. Math. (Beograd) (N.S.) **11 (25)** (1971), 135–141.

[2] D. M. Cvetković and I. Gutman, *The new spectral method for Determining the number of spaninng trees*, Publ. Inst. Math. (Beograd) (N.S.) **29 (43)** (1981), 49–52.

[3] E. J. Farrell, *On a general class of graph polynomials*, J. Combinatorial Theory, Ser. B, **26** (1979), 111–122.

[4] E. J. Farrell, *On a class of polynomials obtained from the circuits in a graph and its application to characteristic polynomials of graphs*, Discrete Math. **25** (1979), 121–133.

[5] E. J. Farrell and J. C. Grell, *On circrcit polynomials and detercninants of matrices*, submitted.

[6] F. Harary, C. King, A. Mowshowitz and R. C. Read, *Cospectral graphs and digraphs*, Bull. London Math. Soc. **3** (1971), 321–328.

[7] H. Hutschenreuter, *Einfacher Beweis des Matrix-Gerust-Stazes der Netzwerktheorie*, Wiss. Z. Techn. Hochsch. Illmenau **13** (1967), 403–404.

[8] G. Kirchhoff, *Uber die Auflösung der Gleichungen, auf welche man bei der Unlersuchung der linearen Verteilung galvanischer ströme geführt wird*, Ann. Phys. Chem. **72** (1847), 497–508.

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