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SOME RELATIONS FOR GRAPHIC POLYNOMIALS

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Abstract. Let G be a graph and A and B its two subgraphs with disjoint vertex sets. A number of results is obtained, relating the characteristic, matching and μ-polunomials of G, G-A, G-B and G-A-B.

Introduction. In the present paper we shall consider simple graphs without loops and multiple edges, and three polynomials associated with them. These are the characteristic [**2**], the matching [**1,3**] and the μ-polynomial [**5**]. They will be denoted by $\varphi(G)$, a $\alpha(G)$ and $\mu(G)$, respectively with G standing for the corresponding graph.

Let G be a graph with n vertices, $v_1, v_2, \dots, v_n$. Its adjacency matrix $\mathbf{A}$ is square matrix of order n whose element in the i-th row and j-th column is equal to one if the vertices v_i and v_j are adjacent, and is equal to zero otherwise. The characteristic polynomial of $\mathbf{A}$ is called the characteristic polynomial of the respective graph [**2**]. Hence, if $\mathbf{I}$ is the unit matrix of order n then $\varphi(G) = \varphi(G, x) = \det(x\,\mathbf{I} - boldA)$.

Denoting by $m(G, k)$ the number of selections of k independent edges of the graph G (i. e. the number of its k-matchings), the matching polynomial of G is defined as [**1,3**]

$$\alpha(G) = \alpha(G, x) = x^n + \sum_{k=1}^{n/2} (-1)^k m(G, k) x^{n-2k}.$$

If the graph G is acyclic, then by definition, $\mu(G) = \alpha(G)$. Since the characteristic and the matching polynomial of an acyclic graph coincide [**1,3,4**], in this case we also have $\mu(G) = \varphi(G)$.

In order to define the μ-polynomial of a cyclic graph, suppose that G possesses $r\,(r > 0)$ circuits $Z_1, \dots, Z_r$, and associate a parameter t_i with Z_i, $i = 1, \dots, r$.

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Then [**5**]

$$\mu(G)=\alpha(G)+2\sum_i t_i\alpha(G-Z_i)+4\sum_{i<j} t_it_j\alpha(G-Z_i-Z_j)\\ -\cdots+(-2)^r t_1t_2\dots t_r\alpha(G-Z_1-Z_2-\cdots-Z_r) \tag{1}$$

with the following conventions:

(a) lf among the circuits $Z_{i_1},Z_{i_2},\dots,Z_{i_k}$ of G at least two of them possess at least one common vertex, then $\mu(G-Z_{i_1}-Z_{i_2}-\cdots-Z_{i_k})\equiv 0$.

(b) If the circuits $Z_{i_1},Z_{i_2},\dots,Z_{i_k}$ embrace all the vertices of G, then $\mu(G-Z_{i_1}-Z_{i_2}-\cdots-Z_{i_k})\equiv 1$.

The μ-polynomial is a generalization of both the matching and the characteristic polynomial. From (1) it is evident that for $t_1=t_2=\cdots=t_r=0$, $\mu(G)$ reduces to a $\alpha(G)$. It can be shown [**5**] that for $t_1=t_2=\cdots=t_r=1$, $\mu(G)$ coincides with $\varphi(G)$.

The concept of the μ-polynomial was developed in connection with some problems of theoretical chemistry. The chemical applications of the μ-polynomial are elaborated in [**5**], where a number of its basic properties has also been established. Among them we shall need the following three.

If the graph G is composed of components $G_1,G_2,\dots,G_c$, then we shall write $G=G_1\dot{+}G_2\dot{+}\cdots\dot{+}G_c$.

LEMMA 1. $\mu(G_1\dot{+}GZ\dot{+}\cdots\dot{+}G_c)=\mu(G_1)\mu(G_2)\dots\mu(G_c)$.

LEMMA 2. *Let G be an arbitrary graph and u its vertex: Then*

$$\mu(G)=x\mu(G-u)-\sum_j\mu(G-u-v_j)-2\sum_k t_k\mu(G-Z_k). \tag{2}$$

The first summation on the r. h. s. of (2) goes over all vertices v_j which are adjacent to u; the second summation goes over all circuits Z_k which contain the vertex u.

LEMMA 3. *Let e be an edge of G, connecting the vertices u and v. If e does not belong to any circuit of G, then $\mu(G)=\mu(G-e)-\mu(G-u-v)$.*

For the characteristic and matching polynomial of a graph and some of its subgraphs two peculiar relations hold.

LEMMA 4. *If G is a graph and u and v are two distinct vertices, of G then*

$$\varphi(G-u)\varphi(G-v)-\varphi(G)\varphi(G-u-v)=\left[\sum_i\varphi(G-P_i)\right]^2 \tag{3}$$

$$\alpha(G-u)\alpha(G-v)\alpha(G)\alpha(G-u-v)=\sum_i[\alpha(G-P_i)]^2 \tag{4}$$

In both expressions P_i denotes a path and the summations go over all paths in G, which connect the vertices u and v.

Formula (3) is a graph-theoretical reinterpretation of a long-known result for determinants [**7**], whereas (4) was discovered by Heilmann and Lieb [**6**].

As a matter of fact, in the theory of determinants the following result of Jacobi from 1833 is known [**7**, Theorem 1.5.3]. Let

$$D = \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{vmatrix}$$

be a determinant of order n. Let M be a k-rowed minor of D, M^* the corresponding minor of the adjungate of D and $\tilde{M}$ the cofactor of M in D. Then $M^* = D^{k-1}\tilde{M}$. For $k = 2$ we get as a special case of the above equation

$$\begin{vmatrix} A_{uu} & A_{uv} \\ A_{vu} & A_{vv} \end{vmatrix} = D \cdot D_{uv,uv}$$

where A_{uv} is the cofactor of the element a_{uv} and $D_{uv,uv}$, is the determinant of order $n-2$ obtained when the r-th and the s-th rows and columns are deleted from D. This yields $A_{uu}A_{vv} - D \cdot D_{uv,uv} = (A_{uv})^2$.

Suppose now that D is equal to det $(xI - A)$. Then from the definition of the characteristic polynomial of a graph, we immediately have $D = \varphi(G)$, $A_{uu} = \varphi(G-u)$, $A_{vv} = \varphi(G-v)$ and $D_{uv,uv} = \varphi(G-u-v)$. The fact that

$$A_{uv} = \sum_i \varphi(G, P_i)$$

is just another formulation on Coates' formula [**2**, p. 47].

The main results. In this section we report some relations for the μ-polynomial, whose form is similar to that of eqs. (3) and (4). The following two theorems and their corollaries are our main results.

Let A, B, X and Y be rooted graphs. Let H be another graph and u and v two distinct vertices of H. Construct the graph G by identifying the roots of A and X with u, and by identifying the roots of B and Y with v (Fig. 1).

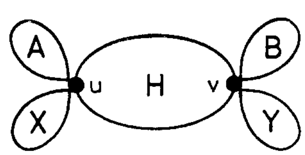

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Fig. 1

THEOREM 1. *Let A', B', X' and Y' denote the subgraphs obtained by deleting the rooted vertex from A, B, X and Y, respectively. Then,*

$$\mu(G-A)\mu(G-B)-\varphi(G)\varphi(G-A-B)=\mu(A')\mu(B')\mu(X')\mu(Y') \\ [\mu(H-u)\mu(H-v)-\mu(H)\mu(H-u-v)]. \tag{5}$$

COROLLARY 1.1.
$\varphi(G-A)\varphi(G-B)-\varphi(G)\varphi(G-A-B)=[\varphi(A')\varphi(B')]^{-1}\left[\sum_i \varphi(G-P_i)\right]^2$.

COROLLARY 1.2.
$\alpha(G-A)\alpha(G-B)-\alpha(G)\alpha(G-A-B)=[\alpha(A')\alpha(B')]^{-1}\sum_i[\alpha(G-P)]^2$.

COROLLARY 1.3.
$\alpha(G-A)\alpha(G-B)-\alpha(G)\alpha(G-A-B)=\sum_i \alpha(G-A-P_i)\alpha(G-B-P_i)$.

The summations in Corollaries 1.1 – 1.3 go over all paths P_i of the graph G, connecting the vertices u and v.

THEOREM 2. *Let H be a graph and u and v two distinct vertices of H. If u and v are connected by a unique path P, then*

$$\mu(H-u)\mu(H-v)-\mu(H)\mu(H-u-v)=[\mu(H-P)]^2. \tag{6}$$

COROLLARY 2.1. *If the vertices u and v of the graph G (from Theorem 1) are connected by a unique path P, then*

$$\mu(G-A)\mu(G-B)-\mu(G)\mu(G-A-B)=\mu(G-A-P)\mu(G-B-P).$$

Proof. In order to prove Theorem 1 we need an auxiliary result.

LEMMA 5. *Let $R_1, R_2, \ldots, R_m$ be routed graphs and $u_1, u_2, \ldots, u_m$, the corresponding roots. Construct the graph R by identifying the roots of all R_i, $i=1,2,\ldots,m$. The vertex so obtained will be denoted by u. Then*

$$\mu(R)=\mu(R_1)\mu(R_2')\ldots\mu(R_m')+\mu(R_1')\mu(R_2)\ldots\mu(R_m')+ \\ +(R_1')\mu(R_2')\ldots\mu(R_m)-(m-1)x\mu(R_1')\mu(R_2')\ldots\mu(R_m') \tag{7}$$

where $R_i'=R_i-u_i$, $i=1,2,\ldots,m$.

Proof. Since the vertex u is a cutpoint in R, it cannot happen that a circuit of R lies partially in R_i and partially R_j, $i \neq j$. Then applying Lemma 2 we get

$$\mu(R)=x\mu(R-u)-\sum_{i=1}^{m}\sum_{j_i}\mu(R-u-v_{j_i})-2\sum_{i=1}^{m}\sum_{k_i}t_{k_i}\mu(R-Z_{k_i}) \tag{8}$$

where v_{j_i} denotes a vertex of R_i which is adjacent to u_i and Z_{k_i} denotes a circuit of R_i which contains the vertex u_i; the appropriate summations go over all vertices v_{j_i} and all circuits Z_{k_i}, respectively.

From the construction of the graph R it is evident that

$$\begin{aligned} R-u &= R'_1 \dotplus R'_2 \dotplus \cdots \dotplus R'_m \\ R-u-v_{j_i} &= R'_1 \dotplus \cdots \dotplus R'_i - v_{j_i} \dotplus \cdots \dotplus R'_m \\ R-z_{k_i} &= R'_1 \dotplus R'_2 \dotplus \cdots \dotplus R'_m \\ R-u-v_{j_i} &= R'_1 \dotplus \cdots \dotplus R_i - z_{k_i} \dotplus \cdots \dotplus R'_m \end{aligned}$$

and bearing in mind Lemma 1 we transform (8) into

$$\mu(R) = \\ x\prod_{h=i}^{m}\mu(Rh') - \sum_{i=1}^{m}\prod_{h\neq i}\mu(Rh')\left[\sum_{j_i}\mu(R_i-u_i-v_{j_i}) + 2\sum_{k_i}t_{k_i}\mu(R_i-Z_{k_i})\right] \tag{9}$$

On the other hand, application of Lemma 2 to R_i gives

$$\mu(R_i) = x\mu(R_i-u_i) - \sum_{j_i}\mu(R_i-u_i-v_{j_i}) - 2\sum_{k_i}t_{k_i}\mu(R_i-Z_{k_i})$$

which combined with (9) gives

$$\mu(R) = x\prod_{h=1}^{m}(Rh') + \sum_{i=1}^{m}\prod_{h\neq i}\mu(Rh')[\mu(R_i) - x\mu(R'_i)].$$

Formula (7) follows now immediately. □

Proof of Theorem 1. Lemma 5 can, of course, be used for all graphs possessing cutpoints. Since the vertices u and v of the graph G are cutpoints (see Fig. 1) we may apply formula (7) to G and its subgraphs $G-A$ and $G-B$.

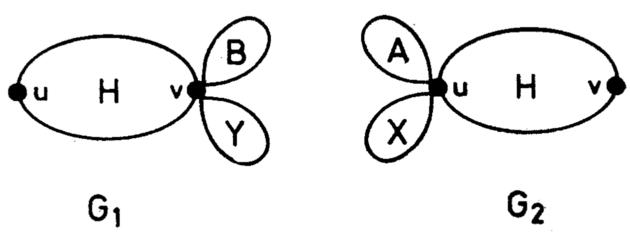

Fig. 2

Defining the graph G_1 as obtained by identifying the roots of B and Y with the vertex v of H (see Fig. 2), we arrive at the following special case of (7):

$$\mu(C) = \mu(A)\mu(X')\mu(G_1-u) + \mu(A')\mu(X)\mu(G_1-u) + \mu(A')\mu(X')\mu(G_1) - \\ -2x\mu(A')\mu(X')\mu(G_1-u). \tag{10}$$

Let the graph G_2 be obtained, in analogy to G_1, by identifying the roots of A and X with the vertex u of H (see Fig. 2). Then we immediately conclude that $G-A = X' \dot{+} (G_1-u)$, $G-B = Y' + (G_2-v)$ and $G-A-B = X' \dot{+} Y' \dot{+} (H-u-v)$ and therefore

$$\mu(G-A) = \mu(X')\mu(G_1-u), \quad \mu(G-B) = \mu(Y')\mu(G_2-v)$$
$$\mu(G-A-B) = \mu(X')\mu(Y')mu(H-u-v).$$

On the other hand, by Lemma 5,

$$\mu(G_1) = \mu(B)\mu(Y')\mu(H-v) + \mu(B')\mu(Y)\mu(H-v) + \mu(B')\mu(Y')\mu(H) - \\ -2x\mu(B')\mu(Y')\mu(H-v) \tag{11}$$

$$\mu(G_1-u) = \mu(B)\mu(Y')\mu(H-u-v) + \mu(B')\mu(Y)\mu(H-u-v) + \\ \mu(B')\mu(Y')\mu(H-u) - 2x\mu(B')\mu(Y')\mu(H-u-v) \tag{12}$$

$$\mu(G_2-v) = \mu(A)\mu(X')\mu(H-u-v) + \mu(A')\mu(X)\mu(H-u-v) + \\ \mu(A')\mu(X')\mu(H-v) - 2x\mu(A')\mu(X')\mu(H-u-v). \tag{13}$$

Substituting eqs. (10)–(13) into the 1. h. s. of formula (5) we obtain its r. h. s. after a lengthy calculation. □

Corollary 1.1 follows for $t_1 = t_2 = \cdots = t_r = 1$, by taking into account eq. (3) and the fact that $G-P_i = A' \dot{+} B \dot{+} X \dot{+} Y \dot{+} (H-P_i)$. Corollary 1.2 is obtained in a similar manner for $t_1 = t_2 = \cdots = t_r = 0$ using eq. (4). Corollary 1.3 is based on the fact that because of $(G-A-P_i) \dot{+} (G-B-P_i) = A' \dot{+} B' \dot{+} X' \dot{+} Y' \dot{+} (H-P_i) \dot{+} (H-P_i)$, we have

$$\alpha(A')\alpha(B')\alpha(X')\alpha(Y')\alpha(H-P_i)^2 = \alpha(G-A-P_i)\alpha(G-B-P_i). \tag{14}$$

Proof of Theorem 2 will be performed by induction on the length p of the path P.

Let $H_0, H_1, \ldots, H_p$ be rooted graphs whose roots are denoted by $v_0, v_1, \ldots, v_p$, respectively. Then the graph H (from Theorem 2) can be constructed by joining the vertices v_{i-1} and v_i by a new edge e_i, $i = 1, \ldots, p$ (see Fig. 3). In this notation, $v_0 \equiv u$ and $v_p \equiv v$.

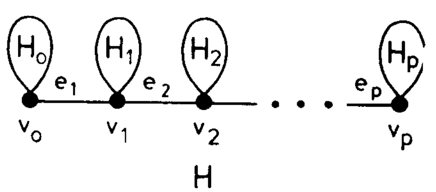

Fig. 3

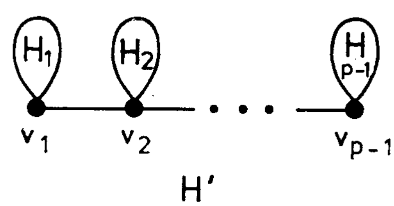

Fig. 4

One should observe that the edges e_i cannot belong to circuits, and thus Lemma 3 is applicable to them.

For $p = 0$, eq. (6) is fulfilled in a trivial maner since then $u \equiv v$ and, by definition, $\mu(H-u-v) \equiv 0$.

For $p = 1$, Lemma 3 gives $\mu(H) = \mu(H_0)\mu(H_1) - \mu(H_0-u)\mu(H_1-v)$ and since

$$\mu(H-u) = \mu(H_0-u)\mu(H_1), \quad \mu(H-v) = \mu(H_0)\mu(H_1-v),$$
$$\mu(H-u-v) = \mu(H-P)\mu(H_0-u)\mu(H_1-v)$$

one immediately verifies that (6) holds.

Suppose now that $p > 1$ and that (6) holds for the graph H' and its vertices v_1 and v_{p-1} (see Fig. 4). For convenience we shall write $v_1 = u'$, $v_{p-1} = v'$. Applying Lemma 3 to the edges e_1 and e_p of H and using Lemma 1, we arrive at

$$\mu(H) = \mu(H_0)\mu(H_p)\mu(H') - \mu(H_0-v_0)\mu(H_p)\mu(H'-u')-$$
$$-\mu(H_0)\mu(H_p-v_p)\mu(H'-v') + \mu(H_0-v_0)\mu(H_p-v_p)\mu(H'-u'v').$$

In addition to this,

$$\mu(H-u) = \mu(H_0-v_0)[\mu(H_p)\mu(H') - \mu(H_p-v_p)\mu(H'-v')],$$
$$\mu(H-v) = \mu(H_p-v_p)[\mu(H_0)\mu(H') - \mu(H_0-v_0)\mu(H'-u')],$$
$$\mu(H-u-v) = (H_0-v_0)\mu(H_p-v_p)\mu(H').$$

Substituting all these relations into the l. h. s. of eq. (6) one obtains

$$\mu(H-u)\mu(H-v) - \mu(H)\mu(H-u-v) =$$
$$\mu(H_0-v_0)^2\mu(H_p-v_p)^2[\mu(H'-u')\mu(H'-v') - \mu(H')\mu(H'-u'-v')].$$

According to the induction hypothesis,

$$\mu(H'-u')\mu(H'-v') - \mu(H')\mu(H'-u'-v') - [\mu(H'-P`)]^2$$

where P' is the (unique) path connecting v_1 and v_{p-1} in H'. Bearing in mind that $H'-P' = (H_1-v_1) \dot{+} (H_2-v_2) \dot{+} \cdots \dot{+} (H_{p-1}-v_{p-1})$ we conclude that

$$\mu(H-u)\mu(H-v) - \mu(H)\mu(H-u-v) =$$
$$= [\mu(H_0-v_0)\mu(H_1-v_1)\mu(H_2-v_2)\ldots\mu(H_p-v_p)]^2$$

which is equivalent to eq. (6). This proves Theorem 2. □

Corollary 2.1 is obtained by combining Theorems 1 and 2 and by taking into account a formula analogous to (14) which holds for the μ-polynomial.

Discussion. It see that Theorems 1 and 2 are special cases of a more general result, which, however remains still to be discovered. We conjecture the following relation for the matching polynomial.

Let G be a graph and A and B its two subgraphs, such that A and B have disjoint vertex sets. Let $P_1, P_2, \ldots, P_s$ be the paths in G whose one endpoint belongs to A, the other endpoint to B, and no other vertex belongs to either A or B. then

$$\alpha(G-A)\alpha(G-B)-\alpha(G)\alpha(G-A-B)=\sum_i \alpha(G-A-P_i)\alpha(G-B-P_i)-$$
$$-\sum_{i<j}\alpha(G-A_i-P_i-P_j)\alpha(G-B-P_i-P_j)+\cdots+ \tag{15}$$
$$+(-1)^{s-1}\alpha(G-A-P_1-P_2-\cdots-P_s)\alpha(G-B-P_1-P_2-\cdots-P_s)$$

where the convention is that whenever at least two among the paths $P_{i_1}, P_{i_2}, \ldots, P_{i_k}$ have at least one common vertex, then $\alpha(G-A-P_{i_1}-P_{i_2}-\cdots-P_{i_k}) \equiv \alpha(G-B-P_{i_1}-P_{i_2}-\cdots-P_{i_k}) \equiv 0$.

If both A and B are one-vertex graphs, then (15) reduces to (4). Another special case of eq. (15), namely when only B is a one-vertex graph, reads

$$\alpha(G-A)\alpha(G-v)-\alpha(G)\alpha(G-A-v)=\sum_i \alpha(G-A-P_i)\alpha(G-v-P_i)$$

and has been established previously [**6**]. Corollary 1.3 is a third special case of the formula (15).

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