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A NOTE ON A BERMOND'S CONJECTURE

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Abstract. If $n \geq 2$ is prime and $k \leq n$, then the arcs of K_n^* can be partitioned into k-cycles iff $n(n-1) \equiv 0 \pmod{k}$.

Let n and k be two non-negative integers. We denote by K_n^* the complete symmetric digraph (directed graph) on n vertices, having $n(n-1)$ arcs, i.e., every ordered pair of vertices is joined by exactly one arc. By a it k-cycle, we mean an elementary cycle (directed cycle) of length k. A *packing* is a set of arc-disjoint cycles of the digraph. A *covering* is a set of cycles covering all the arcs of a digraph. If a digraph has a packing which is also a covering, we say that the arcs of the digraph can be *partitioned* into cycles.

From [**1**] he have the conjecture 4.3.1 due to Bermond: "*The arcs of K_n^* can be partitioned into k-cycles iff $n(n-1) \equiv 0 \pmod{k}$*". If $n \geq 2$ is prime and $k \leq n$, then the conjecture is true, i.e., we have the following

THEOREM. *If $n \geq 2$ is prime and $k \leq n$, then the following are equivalent:* (a) *The arcs of K_n^* can be partitioned into k-cycles;* (b) $n(n-1) \equiv 0 \pmod{k}$.

Proof. The case $n = 2$ is trivial. So, let us suppose $n \geq 3$. Obviously, (a) implies (b), since a necessary condition for the existence of a partition into k-cycles of K_n^* is that the number $n(n-1)$ to be divisible by k. Now, we prove the converse. Because $n \geq 3$ and n is prime, then n is odd. If $k = n$, the theorem follows by [**1**, Theorem 4.1.4]. So, let $k < n$. Then, according to (b), k divides $n-1$ since n is prime. Let $\mathbf{F}$ be a finite field with n elements (e.g., $GF(n)$), and $1_{\mathbf{F}}$ the multiplicative identity of $\mathbf{F}$. Since the multiplicative group of $\mathbf{F}$ contains $n-1$ elements and k divides $n-1$, then there exists $g \in \mathbf{F}$ of order k, i.e., $g^k = 1_{\mathbf{F}}$ and $g \neq 1_{\mathbf{F}}$. We shall identify the vertices of K_n^* with the elements of $\mathbf{F}$ and, for an arbitrary arc (x, y) of K_n^*, we define the following sequence of vertices:

$$x_i = x + (y-x)(g^i - 1_{\mathbf{F}})/(g - 1_{\mathbf{F}}), \ \ i = 1, 2, \ldots, k. \tag{1}$$

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Obviously, $g^i = g^j$ iff $i \equiv j \pmod{k}$. Therefore, the sequence

$$C(k, x, y) = (x, y = x_1, x_2, \ldots, x_{k-1}, x_k = x)$$

is a k-cycle of K_n^*. Let $C(k, x', y') = (x', y' = x'_1, x'_2, \ldots, x'_{k-1}, x'_k = x')$ be another k-cycle of K_n^*, obtained according to (1), such that $C(k, x, y)$ and $C(k, x, y')$ have an arc in common, i.e., $(x_i, x_{i+1}) = (x'_j, x'_{j+1})$. It follows that

$$x_i = x'_j \text{ and } x_{i+1} = x'_{j+1}. \tag{2}$$

From (1), by straightforward calculus, we obtain

$$x_{i+2} - x_{i+1} = g(x_{i+1} - x_i), \tag{3}$$

$$x'_{j+2} - x'_{j+1} = g(x'_{j+1} - x'_j). \tag{4}$$

Thus, from (2) – (4), we have $x_{i+2} = x'_{j+2}$, and, by recurrence, we obtain $x_{i+t} = x'_{j+t}$, $0 \le t \le k-1$ (the indices are taken modulo k). Hence,

$$C(k, x, y) = C(k, x', y'). \tag{5}$$

Considering the set of k-cycles obtained according to (1) for all the possible choices of the arc (x, y), and having in view (5), we obtain a partition of K_n^*, by taking all the distinct k-cycles. Thus, the theorem is completely proved.

REFERENCES

[1] J. C. Bermond and C. Thomassen, *Cycles in diagraphs — A survey*, J. Graph Theory **5** (1981), 1–43.

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