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A NOTE ON THE INDEPENDENCE NUMBER OF AN IDENTICALLY SELF-DUAL PERFECT MATROID DESIGN

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Abstract. Let E be a finite set and $M(E,r)$ an identically self-dual perfect matroid design on E, with hyperplane cardinality $c(M)$, and r as a rank function. If M is not the $r(E)$-uniform matroid, we show that its independence number equals $c(M)-1$.

Introduction. Our matroid-theoretic terminology is based essentially on the books of Randow [**1**] and Welsh [**2**]. Throughout, E will denote an n-set, $|S|$ denotes the cardinality of the set S, and $M{:=}\,M(E,r)$ a *matroid* on E with r as a *rank function*. A subset $s \subseteq E$ is called *independent* if $r(S)=|S|$, a *basis* of M being a maximal independent subset of E. A subset $S \subseteq E$ is called *dependent* if $r(S)<|S|$, a *circuit* of M being a minimal dependent subset of E. It is well-known that a subset is independent iff does not contain a circuit. If $\{B_i\}$ is the set of bases of M, then $\{E-B_i\}$ is the set of bases of the *dual-matroid* M^* on E. If r^* denotes the rank function of M^*, then the following holds:

$$r(E)+r^*(E)=|E|. \tag{1}$$

A *cocircuit* of M is a circuit of M^*, and a hyperplane of M is a complement of a cocircuit in E. A matroid M on E is *identically self-dual* if $M=M^*$, i.e., every circuit is a cocircuit and vice versa.

The *span* (*closure*) S of a subset $\overline{S}$ E is defined by

$$\overline{S}=\{e\in E : r(S\cup\{e\})=r(S)\},$$

and S is called a *closed set* or *flat*, if $S=\overline{S}$. An m-*flat* is defined as a closed subset of E having rank m.

A *perfect matroid design* is a matroid in which each m-flat has a common cardinal, $1\le m\le r(E)$; in particular, all of its hyperplanes have the same cardinality, which we denote by $c(M)$. If $k(M)$ is the minimum cardinality of a circuit

AMS Subject Classification (1980): Primary 05B35 of the matroid M, it follows that $k(M)-1$ is the maximum integer t such that every t-subset of E is independent; $k(M)-1$ is called the *independence number* of M and will be denoted by $i(M)$.

The main result. Let M be a matroid on E and M^* its dual matroid. The family of independent sets of M^* is

$$\{S \subseteq E : r(E-S) = r(E)\} \tag{2}$$

The *k-uniform* matroid on E is defined by taking the family of bases to be $\{S \subseteq E : |S| = k\}$.

Let t be an integer greater than 1. A $t-(v,k,\lambda)$ *design* (V,β) is a v-set V and a system β of k-subsets of $V(k<v)$, called blocks, such that every t-subset of V is contained in exactly λ blocks (repeated blocks are admissible in the system β). For example, $1-(4,2,1)$, $3-(8,4,1)$ and $5-(12,6,1)$ are identically self dual perfect matroid designs. Moreover, no one of these is an $r(E)$-uniform matroid. Thus, the following theorem seems to be of interest.

THEOREM. *Let M be an identically self dual perfect matroid design. If M is not the $r(E)$-uniform matroid, then $i(M) = c(M) - 1$.*

Proof. From [**2**, Theorem 6, p. 212] we have

$$c(M) \le n/2. \tag{3}$$

Since M is identically self-dual, then, by (1), we obtain

$$r(E) = n/2. \tag{4}$$

On the other hand, for each hyperplane H of M, the following holds:

$$r(E) - 1 = r(H) \le |H| = c(M). \tag{5}$$

Hence, from (3) – (5), it follows that

$$n/2 - 1 \le c(M) \le n/2. \tag{6}$$

Suppose that $c(M) = n/2-1$. Thus, by (4) and (5), it follows that every hyperplane is an independent set, i.e., by (2), for each hyperplane H we have

$$r(E-H) = r(E). \tag{7}$$

Now, let $S \subseteq E$ be such that $|S| = r(E)$. If S is not an independent set, then S contains a circuit, i.e., since M is identically self dual, there exists a hyperplane H such that

$$E - H \subseteq S. \tag{8}$$

From (7) and (8), since $r(S) \le r(E)$, we obtain $r(E) \le r(s) \le r(E)$, i.e., $r(S) = |S|$. Thus, S is an independent set. Moreover, because $|S| = r(E)$, it follows that S is a basis of M. Consequently, every subset of cardinality $r(E)$ is a basis of M, i.e., M is the $r(E)$-uniform matroid, contrary to the hypothesis. Hence, in (6), we must have

$$c(M) = n/2. \tag{9}$$

Let C be an arbitrary circuit of M (C is at the same time a cocircuit of M). By (9), we have $|C| = |E| - c(M) = c(M)$, i.e., all the circuits of M have cardinality $c(M)$, and therefore, every $[c(M)-1]$-subset of M is independent. Thus, $i(M) = c(M)-1$.

COROLLARY 1. *Let S be a subset of E. If $\overline{S}$ is a hyperline of M (M being like as in the Theorem), and S is not independent, then S is closed.*

Proof. Suppose that $S \neq \overline{S}$. Since $S \subseteq \overline{S}$, it follows that $|S| < |\overline{S}| - c(M)$, i.e., $|S| \leq c(M) - 1 = i(M)$. Thus (because every set contained in an independent set is independent) S is independent, contradicting the hypothesis, and the corollary is proved.

COROLLARY 2. *If S a subset of E such that $0 \leq r(S) \leq c(M) - 2$, then S is independent in M (M taken as in Theorem).*

Proof. Suppose that S is not independent, i.e., that there exists a circuit C such that $C \subseteq S$. Thus, $c(M) - 1 = r(C) \leq c(M) - 2$, which is absurd. Hence the corollary is proved.

REFERENCES

[1] R. Randow, *Introduction to the Theory of Matroids*, Lecture Notes in Economics and Mathematical Systems, Springer-Verlag, 1975.

[2] D. J. A. Welsh, *Matroid Theory*, Academic Prees, New York, 1976.

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