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# UNIONS AND INTERSECTIONS OF ISOMORYHIC IMAGES OF NONSTANDARD MODELS OF ARITHMETIC

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**Abstract.** We consider those initial segments of a nonstandard model $\mathfrak{M}$ of Peano arithmetic (abbreviated by $P$) which can be obtained as unions or intersections of initial segments of $\mathfrak{M}$ isomorphic to $\mathfrak{M}$. For any consistent theory $T \supseteq P$ we find models of $T$ having collections of initial segments densely ordered by inclusion so that for any segment $I$ from such collection and any $k \in \omega$ the family $\mathcal{A}_k^{\mathfrak{M}} = \{\mathfrak{N} | \mathfrak{N} \subseteq_e \mathfrak{M}.\mathfrak{N} \prec_{\Sigma_k} \mathfrak{M}, \mathfrak{N} \cong \mathfrak{M}\}$ can be partioned into two disjoint parts $\mathcal{A}_1$, and $\mathcal{A}_2$ satisfying $I = \bigcup \mathcal{A}_1 = \bigcap \mathcal{A}_2$ i.e. $I$ is a "point of accumulation" for all families $\mathcal{A}_k^{\mathfrak{M}}$. We investigate. the order type of such collections of segments in the case of recursively saturated models of $P$.

We denote nonstandard. models of $P$ by $\mathfrak{M}$, $\mathfrak{N}$ and $\mathfrak{K}$ and their domains by $|\mathfrak{M}|$, $|\mathfrak{N}|$ and $|\mathfrak{K}|$, respectively; $L_p$ denotes the language of $P$, $\mathcal{N}$ denotes the structure of natural numbers and $\omega$ stands for its domain. If $\mathfrak{M}$ is a model of $P$ and $\mathfrak{N}$ a structure for $L_p$ such that $\mathfrak{N} \subseteq \mathfrak{M}$, then by $\overline{\mathfrak{N}}$ we denote the smallest initial segment of $\mathfrak{M}$ containing $\mathfrak{N}$; $\mathfrak{N} \subseteq_e \mathfrak{M}$ ($\mathfrak{N} \prec_e \mathfrak{M}$) means that $\mathfrak{M}$ is an end extension (elementary end extension) of $\mathfrak{N}$, while $\mathfrak{N} \prec_{\Sigma_k} \mathfrak{M}$ means that for all $\Sigma_k$ formulas $\varphi$ and all $a_1, \ldots, a_n \in |\mathfrak{M}|$, $\mathfrak{N} \models \varphi[a_1, \ldots, a_n]$ holds iff $\mathfrak{M} \models \varphi[a_1, \ldots, a_n]$ holds. For any $\mathfrak{M} \models P$ and $a \in \mathfrak{M}$, let $I_a = \{b \in |\mathfrak{M}| \,|\, b < a\}$. We use the consequence of Matijasevič's theorem asserting that for any models $\mathfrak{M}$, $\mathfrak{N}$ of $P$, $\mathfrak{M} \subseteq \mathfrak{N}$ implies $\mathfrak{M} \prec_{\Sigma_0} \mathfrak{N}$. Thus, $\mathcal{A}_0^{\mathfrak{M}} = \{\mathfrak{N} | \mathfrak{N} \subseteq_e \mathfrak{M}, \mathfrak{N} \cong \mathfrak{M}\}$. If $\Gamma$ is a set of sentences of $L_p$ then $\mathrm{Th}_\Gamma(\mathfrak{M})$ denotes the set of all sentences from $\Gamma$, which are true in $\mathfrak{M}$. We use te fact that for any models $\mathfrak{M}$, $\mathfrak{N}$ of $P$, $\mathfrak{N} \subset_e \mathfrak{M}$ implies $\mathrm{SSy}(\mathfrak{M}) = \mathrm{SSy}(\mathfrak{N})$. The following hierachical refinement of Gaifman's Splitting Theorem is Theorem 1.2 from [**3**].

PROPOSITION 0.1. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be models of $P$ and $\mathfrak{N} \prec_{\Sigma_k} \mathfrak{M}$. Then $\mathfrak{N} \prec_e \overline{\mathfrak{N}} \subseteq_e \mathfrak{M}$ and $\overline{\mathfrak{N}} \prec_{\Sigma_k} \mathfrak{M}$.*

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The following proposition is a hierarchical generalization of Theorem 2.4 (ii) from [**5**], and can be proved in the same way.

PROPOSITION 0.2. *The following are equivalent:* (i) *for arbitrary* $a \in |\mathfrak{N}|$, $\mathfrak{M}$ *is isomorphic to an initial segment of* $\mathfrak{N}$ $\Sigma_k$*-elementarily embedded in* $\mathfrak{N}$, *which contains* $a$. (ii) $\mathrm{Th}_{\Pi_{k+2}}\mathfrak{M} \subseteq \mathrm{Th}_{\Pi_{k+2}}(\mathfrak{N})$ *and* $\mathrm{SSy}(\mathfrak{M}) = \mathrm{SSy}(\mathfrak{N})$. $\square$

Let us consider those initial segments of a model $\mathfrak{M}$ of $P$ which can be obtained as unions or intersections of initial segments of $\mathfrak{M}$ isomorphic to $\mathfrak{M}$. As it was shown in [**3**], (see also [**1**]) $\bigcap \mathcal{A}_k^{\mathfrak{M}}$ is the smallest initial segment of $\mathfrak{M}$ containing all $\Sigma_{k+1}$-definable points of $\mathfrak{M}$ on the other hand $\bigcup \mathcal{A}_k^{\mathfrak{M}} = \mathfrak{M}$.

LEMMA 1.1. *Let* $I_1$ *and* $I_2$ *be initial segments of a nonstandard model* $\mathfrak{M}$ *of* $P$ *such that* $\omega \subset I_2 \subset I_1$. *Then* $I_1$ *contains a model* $\mathfrak{N}$ *of* $P$ *such that* $\mathfrak{N} \subset_e \mathfrak{M}$, $\mathfrak{N} \prec_{\Sigma_k} \mathfrak{M}$, $\mathfrak{N} \cong \mathfrak{M}$, $\mathfrak{N} \nsubseteq I_2$ *iff it contains a model* $\mathfrak{K} \models P$ *such that* $\mathfrak{K} \prec_{\Sigma_k} \mathfrak{M}$, $\mathrm{Th}_{\Pi_{k+2}}(\mathfrak{K}) \supseteq \mathrm{Th}_{\Pi_{k+2}}(\mathfrak{M})$ *and* $\mathfrak{K} \nsubseteq I_2$.

*Proof*: Suppose that there is a model $\mathfrak{K}$ satisfying the conditions from Lemma 1.1, and let $a \in |\mathfrak{K}| \setminus I_2$. Proposition 0.1 implies $\mathfrak{K} \prec \overline{\mathfrak{K}} \prec_{\Sigma_k} \mathfrak{M}$ and $\overline{\mathfrak{K}} \subset_e \mathfrak{M}$; thus $\mathrm{Th}_{\Pi_{k+2}}(\overline{\mathfrak{K}}) \supseteq \mathrm{Th}_{\Pi_{k+2}}(\mathfrak{M})$ and $\mathrm{SSy}(\overline{\mathfrak{K}}) = \mathrm{SSy}(\mathfrak{M})$ holds. According to Proposition 0.2, model $\mathfrak{M}$ is isomorphic to a submodel $\mathfrak{N}$ of $\overline{\mathfrak{K}}$ such that $\mathfrak{N} \subseteq_e \overline{\mathfrak{K}}$, $\mathfrak{N} \prec_{\Sigma_k} \overline{\mathfrak{K}}$ and $a \in |\mathfrak{N}|$. Since $\mathfrak{N} <_{\Sigma_k} \overline{\mathfrak{K}} <_{\Sigma_k} \mathfrak{M}$ and $a \in \mathfrak{N} \setminus I_2$ imply $\mathfrak{N} \prec_{\Sigma_k} \mathfrak{M}$ and $\mathfrak{N} \nsubseteq I_2$, we conclude that $\mathfrak{N}$ satisfies the conditions from Lemma 1.1. The converse is obvious.

COROLLARY 1.2. *Let* $I$ *be an initial segment of a nonstandard model* $\mathfrak{M}$ *of* $P$ *and* $I \neq \omega$ *Then:*

(i) *There is a subfamily* $\mathcal{A} \subseteq \mathcal{A}_k^{\mathfrak{M}}$ *such that* $I = \bigcup \mathcal{A}$ *iff for all* $a \in I$ *there is a model* $\mathfrak{K}_a \models P$ *suchh that* $a \in \mathfrak{K}_a$, $\mathfrak{K}_a \prec_{\Sigma_k} \mathfrak{M}$ *and* $\mathrm{Th}_{\Pi_{k+2}}(\mathfrak{K}_a) \supseteq Th_{\Pi_{k+2}}(\mathfrak{M})$.
*5* (ii) *There is a subfamily,* $\mathcal{A} \subseteq \mathcal{A}_k^{\mathfrak{M}}$ *such that* $I = \bigcap \mathcal{A}$, *iff* $I \cong \mathfrak{M}$ *or for all* $a \in |\mathfrak{M}| \setminus I$ *there is a model* $\mathfrak{K}_a \subseteq I_a$ *such that* $\mathfrak{K}_a \subseteq I$, $\mathfrak{K}_a \models P$, $\mathfrak{K}_a \prec_{\Sigma_k} \mathfrak{M}$ *and* $\mathrm{Th}_{\Pi_{k+2}}(\mathfrak{K}_a) \supseteq \mathrm{Th}_{\Pi_{k+2}}(\mathfrak{M})$.

*Proof.* (i) We apply Proposition 2.1. to all pairs $I$, $I_a$, $a \in I$.

(ii) $I \cong \mathfrak{M}$, let $\mathcal{A} = \{I\}$; otherwise, we apply Proposition 2.1 to all pairs $I_a$, $I$ where $a \in |\mathfrak{M}| \setminus I$.

The following lemma, which is useful for applications of Corollary 1.2, can be proved easily.

LEMMA 1.3. *Let* $\mathfrak{M}$ *and* $\mathfrak{N}$ *be arbitrary models for the same language. Then* $\mathfrak{N} \prec_{\Sigma_k} \mathfrak{M}$ *implies* $\mathrm{Th}_{\Pi_{k+2}}(\mathfrak{M}) \subseteq \mathrm{Th}_{\Pi_{k+2}}(\mathfrak{N})$. $\square$

COROLLARY. 1.4. *Let* $\mathfrak{M}$ *and* $\mathfrak{N}$ *be nonstandard models of* $P$ *and* $\mathfrak{N} \prec_{\Sigma_{k+1}} \mathfrak{M}$; *then there is a subfamily* $\mathcal{A} \subseteq \mathcal{A}_k^{\mathfrak{M}}$ *such that* $\overline{\mathfrak{N}} = \bigcup \mathcal{A}$.

*Proof.* Immediately from Proposition 0.1, Corollary 1.2 (i) and Lemma 1.3.

As a consequence, we get the following proposition.

PROPOSITION 1.5. *Let* $I$ *be an initial segment of a nonstandard model* $\mathfrak{M}$ *of* $P$ *and* $\mathcal{B}$ *a family of initial segments of* $\mathfrak{M}$ *such that for any* $\mathfrak{N}$ *from* $\mathcal{B}$, $\mathfrak{N} \models P$ *and* $\mathfrak{N} \prec_{\Sigma_{k+1}} \mathfrak{M}$ *holds. Then:*

(i) *If* $I = \bigcup \mathcal{B}$, *then there is a subfamily* $\mathcal{A} \subseteq \mathcal{A}_k^{\mathfrak{M}}$ *such that* $I = \bigcup \mathcal{A}$;

(ii) *If* $I = \bigcap \mathcal{B}$ *and* $I \notin \mathcal{B}$, *then there is a family* $\mathcal{A} \subseteq \mathcal{A}_k$ *such that* $I = \bigcap \mathcal{A}_k^{\mathfrak{M}}$.

PROPOSITION 1.6. *Let* $\mathfrak{M}$ *and* $\mathfrak{N}$ *be nonstandard models of* $P$, *such that* $\mathfrak{N} \subseteq_e \mathfrak{M}$ *and let* $\mathfrak{N}_1, \mathfrak{N}_2, \dots,$ *be a strictly decreasing* $\Sigma_{k+1}$*-elementary chain of initial segments, i.e.* $\mathfrak{N}_i \subseteq_e \mathfrak{M}$, $\mathfrak{N}_1 \succ_{\Sigma_{k+1}} \mathfrak{N}_2 \succ_{\Sigma_{k+1}} \mathfrak{N}_3 \succ_{\Sigma_{k+1}} \dots$, *such that* $\bigcap_{i\in\omega} \mathfrak{N}_i = \mathfrak{N}$. *Then, the family* $\mathcal{A}_k^{\mathfrak{M}}$ *can be divided into two disjoint subfamilies* $\mathcal{A}_k^1$ *and* $\mathcal{A}_k^2$ *such that* $\mathfrak{N} = \bigcup \mathcal{A}_k^1 = \bigcap \mathcal{A}_k^2$.

*Proof.* Since $P$ has definable Skolem functions, the hierachical refinement of the Tarski-Vaught Theorem implies $\bigcap_{i\in\omega} \mathfrak{N}_i = \mathfrak{N} \prec_{\Sigma_{k+1}} \mathfrak{M}$. Thus, letting $\mathcal{A}_k^1 = \{\mathfrak{K} | \mathfrak{K} \in \mathcal{A}_k^{\mathfrak{M}}, \mathfrak{K} \subseteq \mathfrak{N}\}$ and $\mathcal{A}_k^2 = \{\mathfrak{K} | \mathfrak{K} \in \mathcal{A}_k^{\mathfrak{M}}, \mathfrak{K} \supset \mathfrak{N}\}$, we get from Corollary 1.4 and Proposition 1.5 $\mathfrak{N} = \bigcup \mathcal{A}_k^1 = \bigcap \mathcal{A}_k^2$.

COROLLARY 1.7. *Let* $\mathfrak{M}$ *be a nonstandard model of* $P$ *and* $\mathfrak{N}_1 \prec \mathfrak{N}_2 \prec \mathfrak{N}_3 \dots$ *a strictly decreasing elementary chain of initial segments of* $\mathfrak{M}$, *such that* $\bigcap_{i\in\omega} \mathfrak{N}_i \neq \mathcal{N}$. *Then for all* $n \in \omega$ *the family* $\mathcal{A}_k^{\mathfrak{M}}$ *can be divided into two disjoint subfamilies* $\mathcal{A}_k^1$, $\mathcal{A}_k^2$ *such that* $\bigcup \mathcal{A}_k = \bigcap \mathcal{A}_k^2 = \bigcap_{i\in\omega} \mathfrak{N}_i$

*Proof*: Since $P$ has definable Skolem functions, $\bigcap_{i\in\omega} \mathfrak{N}_i \prec \mathfrak{M}$ holds, and consequently, $\bigcap_{i\in\omega} \mathfrak{N}_i \models P$. Since $\bigcap_{i\in\omega} \mathfrak{N}_i \neq \mathcal{N}$, we can apply Proposition 1.6.

We now look for models having such chains.

LEMMA 1.8. *For any consistent extension* $T$ *of* $P$ *there is a countable model* $\mathfrak{M}$ *of* $T$ *having a family of initial segments densely ordered by inclusion such that any member of this family is an intersection of a strictly decreasing elementary chain of initial segments of* $\mathfrak{M}$.

*Proof.* Let $\ll$ be any recursive dense ordering on $\omega$, and $U(x, y, z)$, $V(x, y)$ two new predicate symbols. We consider the theory $T' = T \cup A_1 \cup A_2 \cup A_3 \cup A_4$ of the laguage $L = L_p \cup \{U, V\}$, where $A_1 - A_4$ are defined as follows:

$A_1 = \{\forall x \forall y (V(x, n) \wedge y < x \to V(y, n))$; $n \in \omega$, and the same for $U(x, m, n)\}$;

$A_2 = \{\forall x_1 \dots x_k (V(x_1, n) \wedge \dots \wedge V(x_k, n) \wedge \exists x \varphi(x, x_1, \dots, x_k) \to \exists x (V(x, n) \wedge$
$\varphi(x, x_1, \dots, x_n)))$ for all $k, n \in \omega$, all formulas $\varphi$ of $L_p$, and the same for
$U(x, m, n), m, n \in \omega\}$;

$A_3 = \{\forall x ((V(x, n) \to U(x, n, m)) \wedge (U(x, n, m+1) \to U(x, n, m))) \wedge \exists x (U(x, n, m) \wedge$
$\neg U(x(x, n, m + 1)), n, m \in \omega\}$;

$A_4 = \{\forall x (U(x, n, 1) \to V(x, m))$; for all $m, n \in \omega$ such that $n \gg m\}$.

Theory $T'$ is consistent because any finite subtheory of $T'$ is realized in a model $\mathfrak{M}'$ obtained as a finite chain of elementary end extensions of any model $\mathfrak{M}$ of $T$. Any countable model of $T'$, with the family of initial segments which are interpretations in this model of $U(x, n, m)$ and $\bigcap \mathfrak{M} \in_\omega U(x, n, m), n, m \in \omega$, obviously satisfies the conditions from Lemma 1.8.

Since $T'$ is a recursive theory, the same argument shows that any resplendent countable model of $T$ can be expanded to a model of $T'$ because $T'$ is consistent with $\mathrm{Th}(\mathfrak{M})$ for any $\mathfrak{M}$, $\mathfrak{M} \models T$.

From Lemma 1.8 and Corollary 1.7 the following proposition immediately follows.

PROPOSITION 1.9. *For any consistent extension $T$ of $P$ there is a countable model $\mathfrak{M}$ of $T$ having a collection of initial segments densely ordered by inclusion, so that, for any segment $I$ from the collection and any $k \in \omega$ the family $\mathcal{A}_k^{\mathfrak{M}}$ can he divided into two disjoint parts $\mathcal{A}_k^2$ and $\mathcal{A}_k^2$ so that $I = \bigcup \mathcal{A}_k^1 = \bigcap \mathcal{A}_k^2$.*

Using a Kotlarski's result [**2**] we can prove that in the case of recursively saturated countable models of $P$, we can find such a collection of initial segments of the power $2^\omega$. Namely, in that case, the set $Y = \{\mathfrak{N} \mid \mathfrak{N} \prec_e \mathfrak{M}\}$ is of the order type of Cantor set $2^\omega$ with its lexicographical ordering, and any $\mathfrak{N}$ from $Y$ is isomorphic to $\mathfrak{M}$. We call a pair $(Y_1, Y_2)$ a cut in $Y$ iff $Y_1 \cap Y_2 = \emptyset$, $Y_1 \cup Y_2 = Y$ and for all $I_1, I_2$ from $Y$, $I_1 \in Y_1$ and $I_2 \in Y_2$ implies $I_1 \subseteq I_2$. Since for two different cuts $(I_1, I_2)$ and $(I_1', I_2')$ the sets $\bigcap I_2 \setminus \bigcap I_1$ and $\bigcap I_2' \setminus I_1'$ are disjoint and since $\mathfrak{M}$ is countable, there are only countably many cuts $(Y_1, Y_2)$ such that $\mathfrak{M} \setminus (\bigcap Y_2 \setminus \bigcup Y_1) \neq \emptyset$. It is easy to see that for any cut $(Y_1, Y_2)$ such that $\mathfrak{M} \setminus (\bigcap Y_2 \setminus \bigcup Y_1) = \emptyset$, the segment $I = \bigcap Y_2 = \bigcup Y_1$ satisfies the conditions from Proposition 1.9, and that the family of such segments is of power $2^\omega$ and is densely ordered by inclusion.

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