PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 39 (53), 1986, pp. 13–16

THE ZERO-ONE LAW FOR PRODUCTS OF INTERNAL $*$-FINITELY ADDITIVE PROBABILTTY SPACES

Boško Živaljević

Abstract. A version of the zero-one law for products of internal $*$-finitely additive probability spaces is proved.

1. Introduction. H. J. Keisler proved (see [**2**], [**3**] or [**4**]) that Fubini's theorem is valid for the product of two hyperfinite probability spaces. Namely, if we have two hyperfinite probability spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ we may consider the space $(X \times Y, \mathcal{A} \times \mathcal{B}, \mu \times \nu)$ (here $\mathcal{A} \times \mathcal{B}$ is the internal algebra of all internal subsets of $X \times Y$ on which $\mu \times \nu$ is naturally defined) and the Loeb space $(X \times Y, L(\mathcal{A} \times \mathcal{B}), L(\mu \times \nu))$ for which Fubini's theorem is valid. It is natural to ask if the same theorem is true for the product of lawcountably many hyperfinite probability spaces. In this paper we show that the zero-one law for products of internal $*$-finitely additive probability spaces is true.

The relevant facts about non standard measure theory that are needed in this paper, can be found in [**4**] and we will only repeat some basic definitions. An internal $*$-finitely additive probability space is a triple $(X, \mathcal{A}, \mu)$ where X is an internal set, $\mathcal{A}$ an internal algebra of subsets of X and μ a $*$-finitely additive probability measure on $\mathcal{A}$. Given such a space we may construct (as it was pointed out by P. Loeb) a standard countably additive probability space $(X, L(\mathcal{A}), L(\mu))$ where $L(\mathcal{A})$ is a σ-algebra containing $\mathcal{A}$ and μ a σ-additive complete measure extending $^\circ \circ \mu$, (where $^\circ$ is, as usual, a standard part map). The space $(X, L(\mathcal{A}), L(\mu))$ is call the Loeb space of a space $(X, \mathcal{A}, \mu)$. An example of an internal $*$-finitely additive probability space is a hyperfinite probability space, i.e. a triple $(H, \mathcal{A}, \mu)$ where H is a hyperfinite set, $\mathcal{A}$ the internal algebra of all internal subsets of H and μ an internal $*$-finitely additive probability measure on $\mathcal{A}$.

In this paper we are working in a polysatured enlargement of a standard structure.

AMS Subject Classification (1980): Primary 03H05

2. Results. From two internal $*$-finitely additive probability spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ we may construct the internal $*$-finitaly additive probability space $(X \times Y, \mathcal{A} \times \mathcal{B}, \mu \times \nu)$ where $\mathcal{A} \times \mathcal{B}$ is the least internal algebra containing the sets of the form $E \times F$ $(E \in \mathcal{A}, F \in \mathcal{B})$, on which the measure $\mu \times \nu$ is naturally defined. As mentioned in the introduction J. Keisler proved Fubini's theorem for the space $(X \times Y, L(\mathcal{A} \times \mathcal{B}), L(\mu \times \nu))$. This is a new result because if we consider the usual standard product of the Loeb spaces $(X, L(\mathcal{A})\, L(\mu))$ and $(Y, L(\mathcal{B}), L(\nu))$, denoted by $(X \times Y, L(\mathcal{A}) \ \times L(\mathcal{B}), L(\mu) \times L(\nu))$, then we have $L(\mathcal{A}) \times L(\mathcal{B}) \subseteq L(\mathcal{A} \times \mathcal{B})$ (and the measures $L(\mu) \times L(\nu)$ and $L(\mu \times \nu)$ coincide of $L(\mathcal{A}) \times L(\mathcal{B})$ and the above inclusion can be strict as Hoover's example shows (see [**4**]).

We are now going to investigate a product of infinitely many spaces.

Let $(X_n, \mathcal{A}_n, \mu_n)$ $(n \in N)$ be a sequence of internal $*$-finitely additive probability spaces (where N is, as usual, the set of standard natural numbers). We may extend this sequence to an internal hyperfinite sequence of internal $*$-finitely additive probability spaces

$$(X_1, \mathcal{A}_1, \mu_1), \dots, (X_H, \mathcal{A}_H, \mu_H) \qquad (H \in^* N \setminus N) \tag{1}$$

At the same time we consider standard sequence of Loeb's spaces

$$(X_n, L(\mathcal{A}_n), L(\mu_n)) \qquad (n \in N) \tag{2}$$

In the internal product[1] $\prod_1^H X_k$ we can introduce an internal $*$-finitely additive measure $\prod_1^H \mu_k$, so that we have an internal $*$-finitely additive probability space $\left(\prod_1^H X_k, \prod_1^H \mathcal{A}_k, \prod_1^H \mu_k\right)$, where $\prod_1^H \mathcal{A}_k$ is he least internal algebra containing all sets of the form $E_1 \times \dots \times E_H$ $(E_k \in \mathcal{A}_k, k = 1, \dots, H)$. The corresponding Loeb space is $\left(\prod_1^H X_k, L\left(\prod_1^H \mathcal{A}_k\right), L\left(\prod_1^H \mu_k\right)\right)$.

Considering the standard product of probability spaces (2) we obtain the probability space $(\prod_N X_k, \prod_N L(\mathcal{A}_k), \prod_N L(\mu_k))$ where $\prod_N L(\mathcal{A}_k)$ is the σ-algebra containing all cylinder sets, i.e. sets of the form

$$P_1 \times \dots \times P_n \times X_{n+1} \times \dots \quad (P_k \in L(\mathcal{A}_k), k = 1, \dots, n)$$

and the complete probability measure $\prod_N L(\mu_k)$.

There exist a natural function $ST : \prod_1^H X \to \prod_N X_k$ which assigns to every $*$-finite sequence $(\alpha_1, \dots, \alpha_H) \in \prod_1^H X_k$ its projection on its first N coordinates i.e.

$$ST(\alpha_1, \dots, \alpha_H) = (\alpha_1, \dots, \alpha_n, \dots) \quad (n \in N)$$

From the following lemma we know that ST is onto, so that we can consider the probability space $(\prod_N X_k, \mathcal{A}, m)$ where

$$\mathcal{A} = \left\{ Y \subseteq \prod_{k \in N} X_k : ST^{-1}(Y) \in L\left(\prod_{k=1}^{H} \mathcal{A}_k\right)\right\}$$

[1] If otherwise not stated $\prod_1^H$ and $\prod_N$, will mean $\prod\limits_{k=1}^{H}$ and $\prod\limits_{k \in N}$ respectively.

and m is the measure defined by: $m(Y) = L\left(\prod_1^H \mu_k\right)(ST^{-1}(Y))$.

LEMMA 1. (i) ST *is onto*; (ii) $\prod_N L(\mathcal{A}_k) \subseteq \mathcal{A}$, $\prod_N L(\mu_k)$ *and* m *agree on* $\prod_N L(\mathcal{A}_k)$.

Proof (i) Let $(\alpha_1, \dots, \alpha_n, \dots) \in \prod_N X_k$. We can extend this sequence to a hyperfinite sequence $(\alpha_1, \dots, \alpha_{H_0})$ such that $\alpha_k \in X_k$ for $k = 1, \dots, H_0$. In both cases, $H_0 \geq H$ or $H_0 < H$, we can cut off the given sequence or extend it to the sequence $(\alpha_1, \dots, \alpha_H)$ such that we have

$$ST(\alpha_1, \dots, \alpha_H) = (\alpha_1, \dots, \alpha_n, \dots) \quad (n \in N).$$

(ii) Let $\mathcal{P} = \{E_1 \times \dots \times E_n \times X_{k+1} \times : E_k \in \mathcal{A}_k \, (k = 1, \dots, n;\, n \in N)\}$. First we will prove that the σ-algebra $\prod_N L(\mathcal{A}_k)$ can be obtained using the ordinary Carathéodory extention procedure for the measure $\prod_N L(\mu_k)$ on the semiring $\mathcal{P}$.

If $F_k \in L(\mathcal{A}_k)$ and $L(\mu_k)\,(F_k) = 0$ $(k = 1, \dots, n;\, n \in N)$, then we can pick sets $E_k \in \mathcal{A}_k$ $(k = 1, \dots, n)$ such that $E_k \supseteq F_k$ and $L(\mu_k)(E_k) < \sqrt[n]{\varepsilon}$ (or every standard positive ε), from which it follows that

$$F_1 \times \dots \times F_k \times X_{n+1} \times \dots \subseteq E_1 \times \dots \times E_k \times X_{n+1} \times \dots$$

and $\prod_N L(\mu_k)(E_1 \times \dots \times E_n \times X_{r+1} \times \dots) < \varepsilon$. This shows that $F_1 \times \dots \times F_n \times X_{n+1} \times \dots$ is in Carathéodory's extention $\overline{\mathcal{P}}$ of the semiring $\mathcal{P}$.

In the general case for $F_k \in L(\mathcal{A}_k)$ $(k = 1, \dots, n)$ there exists sets $E_k \in \mathcal{A}_k$ $(k = 1, \dots, n)$ such that $L(\mu_k)(E_k \Delta F_k) = 0$ where Δ is the symmetric difference. We have

$$\begin{aligned}(F_1 \times \dots \times F_n \times X_{n+1} \times \dots)\Delta(E_1 \times \dots \times E_n \times X_{n+1} \times \dots) \subseteq \\ \subseteq \left(F_1 \Delta E_1) \times \prod_{\substack{k \in N \\ k>1}}\right) \cup \left(X_1 \times (F_2 \Delta E_2) \times \prod_{\substack{k \in N \\ k>2}}\right) \cup \dots \\ \dots \cup \left(\prod_{k=1}^{n-1} X_k \times \left(F_n \Delta E_n \times \prod_{\substack{k \in N \\ k>2}} X_k\right)\right).\end{aligned}$$

The sets on the right-hand side of the above inclusion have $\prod_N L(\mu_k)$ measure zero and, because of the completeness of $\prod_N L(\mu_k)$, $F_1 \times \dots \times F_n \times X_{n+1} \times \dots$ belongs to $\overline{\mathcal{P}}$. We have proved

$$\prod_{k \in N} L(\mathcal{A}_k) = \overline{\mathcal{P}} \tag{3}$$

Now we can easily prove the second part of the lemma. For cylindric sets of the form $E_1 \times \dots \times E_n \times X_{n+1} \times \dots = D$ $(E_k \in \mathcal{A}_k, k = 1, \dots, n)$ we have $ST^{-1}(D) = E_1 \times \dots \times E_n \times X_{n+1} \times \dots \times X_H$ and $L\left(\prod_1^H \mu_k\right)(ST^{-1}(D)) = \prod_N L(\mu_k)(D)$, so (ii) holds. But the sets for which (ii) is true from the σ-algebra, so, if we keep in mind (3), (ii) is true for $\prod_N L(\mathcal{A}_k)$.

Let $\mathcal{B}$ be the σ-algebra generated by the sets of the form $E \times X_{n+1} \times \dots$ $(n \in N)$, where E belongs to $\prod_1^n \mathcal{A}_k$, i.e. E is an element of the internal algebra generated by the sets of the form $E_1 \times \dots \times E_n$ $(E_k \in \mathcal{A}_k, k = 1, \dots, n)$.

LEMMA 2. $\mathcal{B} \subseteq \mathcal{A}$.

Proof. $\mathcal{B}$ is generated by the sets of the form $E \times X_{k+1} \times \dots = D$, $E \in \prod_1^n \mathcal{A}_k$; so we have $ST^{-1}(D) = E \times X_{n+1} \times \dots \times X_H \in \prod_!^n \mathcal{A}_k$, and the result follows.

Therefore we have the probability space $(\prod_N X_k, \mathcal{B}, m)$. If we use the same notation as in the proof of Lemma 1, then $\sigma(\mathcal{P}) \subseteq \mathcal{B}$ ($\sigma(\mathcal{P})$ is the least σ-algebra containing $\mathcal{P}$), and the inclusion can be strict (as Hoover's example shows). Thus, the next theorem is not the usual zero-one law.

A set $D \subseteq \prod_N X_k$ is a tail set if and only if it is closed with respect to a change of finitely many coordinates of its elements, i.e. $(\alpha_1, \dots, \alpha_n, \alpha_{n+1}, \dots) \in D$ implies $(\beta_1, \dots \beta_n, \alpha_{n+1}, \dots) \in D$ for every $\beta_1 \in X_1, \dots, \beta_n \in X_n$ and every $n \in N$.

THEOREM. *If $D \in \mathcal{B}$ is a tail set, then $m(D) = 0$ or $m(D) = 1$.*

Proof. Let $\pi_{n,m} : \prod_N X_k \to X_n \times \dots \times X_m$ $(m \geq n)$ be the projection, i.e. $\pi_{n,m}(\alpha_1, \dots, \alpha_n, \dots, \alpha_m, \dots) = (\alpha_n, \dots, \alpha_m)$, $\mathcal{K}_{n,m}$ the internal algebra generated by the sets of the form $E_n \times \dots \times E_m$ $(E_k \in \mathcal{A}_k, k = n, \dots, m; m \geq n)$ $\mathcal{F}_{n,m}$ the σ-algebra generated by the sets of the form $\pi_{n,m}^{-1}(E)$ $(E \in \mathcal{K}_{n,m})$ and let $\mathcal{F}_{n,\infty}$ be $\sigma\left(\bigcup_{m \geq n} \mathcal{F}_{n,m}\right)$. Notice that $\mathcal{F}_{n,m} \subseteq \mathcal{F}_{n,m+1}$. Therefore $\bigcup_{m \geq n} \mathcal{F}_{n,m}$ is an algebre. Finally we put $\mathcal{F} = \bigcup_{n \in N} \mathcal{F}_{n,\infty}$. This is the σ-algebra of tail sets belonging to $\mathcal{B}$.

Let $D \in \mathcal{F}$. Then $D \in \mathcal{F}_{n,\infty}$ i.e. $D \in \sigma\left(\bigcup_{n \in N} \mathcal{F}_{1,n}\right)$, so, there are sets $E_k \in \mathcal{F}_{1,n}$ such that $m(D \Delta E_n) \to 0$. But, $D \in \mathcal{F}_{n+1,\infty}$ and the sets E_n and D are independent, i.e. $m(D \cap E_n) = m(D) \cdot m(E_n)$. From this it follows that $m(D) = m(D)^2$ and hence $m(D)$ must be zero or one.

Open questions: (i) Is the Fubini theorem or the zero-one law true for the σ-algebra $\mathcal{A}$?

(ii) Let $\overline{\mathcal{B}}$ be the completion of $\mathcal{B}$. Are $\overline{\mathcal{B}}$ and $\mathcal{A}$ equal?

REFERENCES

[1] A. N. Širjaev, *Vierojatnost*, Nauka, Moskva, 1980.

[2] H. J. Keisler, *Hyperfinite model theory in Logic Colloquium* '76 (Eds. R. O. Gandy and J. M. E. Hyland), North-Holland, Amsterdam, 1977. pp. 6–110.

[3] H. J. Keisler, *An Infinitesimal Approach to Stochastic Analysis*, Mem. Amer. Math. Soc., to appear.

[4] K. D. Stroyan, J. M. Bayod, *Foundations of Infinitesimal Stochastic Analysis*, North-Holland, Amsterdam, to appear.

[5] K. D. Stroyan, W. A. J. Luxemburg, *Introduction to the Theory of Infinitesimals*, Academic Press, New York, 1976.

Prirodno-matematički fakultet
Odsek za matematiku
Vojvode Putnika 43
71000 Sarajevo
Jugoslavija

(Received 20 121 1984)