

AN APPLICATION OF THE RUSCHEWEYH DERIVATIVES II

Shigeyoshi Owa

Abstract. We introduce the class $R(\alpha)$ of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \geq 0)$$

satisfying the condition

$$\Re\{D^{\alpha+1}f(z)/D^{\alpha}f(z)\} > \alpha/(\alpha+1)$$

for some $(\alpha \geq 0)$ and for all $z \in U = \{z : |z| < 1\}$, where $D^{\alpha}f(z)$ denotes the Hadamard product of $z/(1-z)^{\alpha+1}$ and $f(z)$. The object of the present paper is to prove some distortion and closure theorems for functions $f(z)$ in $R(\alpha)$, and to give the result for the modified Hadamard product of functions $f(z)$ belonging to the class $R(\alpha)$. Furthermore, we determine the radii of starlikeness and convexity of functions $f(z)$ in the class $R(\alpha)$.

1. Introduction

Let A denote the class of functions $f(z)$ of the form

$$(1.1) \quad f(z) = \sum_{k=2}^{\infty} *a_k z^k$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$. We denote by S the subclass of univalent functions $f(z)$ in A , and by S^* and K the subclasses of S whose members are starlike with respect to the origin and convex in the unit disk U , respectively. A function $f(z)$ belonging to the class A is said to be starlike of order β ($0 \leq \beta < 1$) in the unit disk U if and only if

$$(1.2) \quad \Re\{zf'(z)/f(z)\} > \beta \quad (z \in U)$$

for some β ($0 \leq \beta < 1$). Further, a function $f(z)$ belonging to the class A is said to be convex of order β ($0 \leq \beta < 1$) in the unit disk U if and only if

$$(1.3) \quad \Re\{1 + zf''(z)/f'(z)\} > \beta \quad (z \in U)$$

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* \sum stands for $\sum_{k=2}^{\infty}$ unless stated otherwise.

for some β ($0 \leq \beta < 1$). We denote by $S^*(\beta)$ and $K(\beta)$ the subclasses of A whose members satisfy (1.2) and (1.3), respectively. Then, it is well-known that $S^*(\beta) \subset S^*$, $K(\beta) \subset K$ for $0 < \beta < 1$, and that $S^*(0) \equiv S^*$, $K(0) \equiv K$ for $\beta = 0$.

Let $f * g(z)$ denote the Hadamard product of two functions $f(z) \in A$ and $g(z) \in A$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = z + \sum b_k z^k,$$

then

$$f * g(z) = z + \sum a_k b_k z^k.$$

By using the Hadamard product, Ruscheweyh [15] defined

$$D^\alpha f(z) = (z/(1-z)^{\alpha+1}) * f(z) \quad (\alpha \geq 1)$$

which implies that

$$(1.4) \quad D^n f(z) = \{z(z^{n-1}f(z))^{(n)}\}/n!$$

for $n \in N \cup \{0\}$, where $N = \{1, 2, 3, \dots\}$. We note that $D^0 f(z) = f(z)$ and $D^1 f(z) = z f'(z)$. The symbol $D^n f(z)$ was named by Al-Amiri [1] the n -th order Ruscheweyh derivative of $f(z)$. With the notation (1.4), Ruscheweyh [15] introduced the classes K_n of functions $f(z)$ in A satisfying the following condition

$$(1.5) \quad \Re\{D^{n+1}f(z)/D^n(z)\} > 1/2 \quad (z \in U)$$

for $n \in N \cup \{0\}$, and he showed the basic property

$$(1.6) \quad K_{n+1} \subset K_n$$

for each $n \in N \cup \{0\}$, where $K_0 \equiv S^*(1/2)$ and $K_1 \equiv K$. Further, in the notation (1.5) a class K_{-1} can be defined as the class of functions $f(z)$ in A satisfying

$$\Re\{f(z)/z\} > 1/2 \quad (z \in U).$$

Since $K_0 \equiv S^*(1/2) \subset S^*$, Ruscheweyh's result (1.6) implies that $K_n \subset S^* \subset S$ for each $n \in N \cup \{0\}$.

Recently, by using the n -th order Ruscheweyh derivative of $f(z)$, Singh and Singh [18] introduced the subclass R_n of A whose members are characterized by the following condition

$$\Re\{D^{n+1}f(z)/D^n f(z)\} > n/(n+1) \quad (z \in U)$$

for $n \in N \cup \{0\}$. We can immediately see that $R_0 \equiv S^*$ and $R_n \subset K_n$ for each $n \in N$. Hence R_n is a subclass of $S^* \subset S$ for each $n \in N \cup \{0\}$. Further we can observe that $R_{n+1} \subset R_n$ for every $n \in N \cup \{0\}$.

In recent years, many classes defined by using the n -th order Ruscheweyh derivative of $f(z)$ were studied by Al-Amiri ([2], [3]), Bulboaca [4], Goel and Sohi ([6], [7]), and Owa ([10], [11]).

In this paper, we introduce the following classes $R(\alpha)$ by using the symbol $D^\alpha f(z)$.

Definition 1. We say that $f(z)$ is in the class $R(\alpha)$ ($\alpha > 0$), if $f(z)$ defined by

$$(1.7) \quad f(z) = z - \sum a_k z^k \quad (a_k \geq 0)$$

satisfies the condition

$$(1.8) \quad \Re\{D^{\alpha+1}f(z)/D^\alpha f(z)\} > x/(\alpha + 1) \quad (z \in U)$$

for some α ($\alpha \geq 0$).

Since $D^1 f(z) = zf'(z)$ and $D^0 f(z) = f(z)$, we can see that $R(0) \equiv T^*$ for $\alpha = 0$, which is the subclass of S^* consisting of functions $f(z)$ of the form (1.7) and was studied by Silverman [17]. $R(1)$ is the subclass of $K_1 \equiv K$ consisting of functions $f(z)$ of the form (1.7). Further, $R(n)$ is the subclass of R_n consisting of functions $f(z)$ of the form (1.7).

2. Fractional calculus

Many essentially equivalent definitions of the fractional calculus that is, fractional derivatives and fractional integrals, have been given in the literature (cf. e.g., [5, Chapter 13], [8], [9], [14], [16], and [19, P. 28 et seq.]). We find it to be convenient to recall here the following definitions which were used recently by Owa [12].

Definition 2. The fractional integral of order α is defined by

$$D_z^{-\alpha} f(z) = \frac{1}{\Gamma(\alpha)} \int_0^z \frac{f(\zeta) d\zeta}{(z - \zeta)^{1-\alpha}},$$

where $\alpha > 0$, $f(z)$ is an analytic function in a simply connected region of the z -plane containing the origin and the multiplicity of $(z - \zeta)^{-\alpha}$ is removed by requiring $\log(z - \zeta)$ to be real when $(z - \zeta) > 0$.

Definition 3. The fractional derivative of order α is defined by

$$D_z^\alpha f(z) = \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta) d\zeta}{(z - \zeta)^\alpha},$$

where $0 \leq \alpha < 1$, $f(z)$ is an analytic function in a simply connected region of the z -plane containing the origin and the multiplicity of $(z - \zeta)^{-\alpha}$ is removed by requiring $\log(z - \zeta)$ to be real when $(z - \zeta) > 0$.

Definition 4. Under the hypotheses of Definition 3, the fractional derivative of order $(n + \alpha)$ is defined by

$$D_z^{n+\alpha} f(z) = d^n D_z^\alpha f(z) / dz^n,$$

where $0 \leq \alpha < 1$ and $n \in N \cup \{0\}$.

With these definitions, recently, Owa [13] showed the following lemma.

LEMMA. Let the function $f(z)$ be defined by (1.7). Then

$$D^{n+\alpha} f(z) = z \{ D_z^{n+\alpha} (z^{n+\alpha-1} f(z)) \} / \Gamma(n + \alpha + 1)$$

for $0 \leq \alpha < 1$, $n \in N \cup \{0\}$ and $z \in U$.

3. Distortion theorems

By using the lemma, we state and prove

THEOREM 1. Let $n \in N \cup \{0\}$, $0 \leq \alpha < 1$ and the function $f(z)$ be defined by (1.7). Then $f(z)$ is in the class $R(n + \alpha)$ if and only if

$$(3.1) \quad \sum \frac{(n + \alpha k + k) \Gamma(n + \alpha + k)}{(k - 1)!} a_k \leq \Gamma(n + \alpha + 2).$$

Equality holds for functions $f(z)$ given by

$$(3.2) \quad f(z) = z - \frac{\Gamma(n + \alpha + 2)(k - 1)!}{(n + \alpha k + k) \Gamma(n + \alpha + k)} z^k \quad (k \geq 2)$$

Proof. Assume that the inequality (3.1) holds true and let $|z| = 1$. Then, by virtue of the lemma, we obtain that

$$\begin{aligned} \left| \frac{D^{n+\alpha+1} f(z)}{D^{n+\alpha} f(z)} - 1 \right| &= \left| \frac{\sum \frac{(1-k) \Gamma(n+\alpha+k)}{(n+\alpha+1)(k-1)!} a_k z^{k-1}}{\Gamma(n+\alpha+1) - \sum \frac{\Gamma(n+\alpha+k)}{(k-1)!} a_k z^{k-1}} \right| \\ &\leq \frac{\sum \frac{\Gamma(n+\alpha+k)}{(n+\alpha+1)(k-1)!} a_k}{\Gamma(n+\alpha+1) - \sum \frac{\Gamma(n+\alpha+k)}{(k-1)!} a_k} \leq 1/(\alpha+1). \end{aligned}$$

This shows that the values of $D^{n+\alpha+1} f(z)/D^{n+\alpha} f(z)$ lie in a circle centered at $w = 1$ whose radius is $1/(\alpha + 1)$. Consequently, we can see that the function $f(z)$ satisfies the condition (1.8), hence, $f(z) \in R(n + \alpha)$.

For the converse, assume that the function $f(z)$ belongs to the class $R(n + \alpha)$ for $n \in N \cup \{0\}$ and $0 \leq \alpha < 1$. Then we get

$$(3.3) \quad \Re \left\{ \frac{D^{n+\alpha+1} f(z)}{D^{n+\alpha} f(z)} \right\} = \Re \left\{ \frac{\Gamma(n + \alpha + 1) - \sum \frac{\Gamma(n+\alpha+k+1)}{(n+\alpha+1)(k-1)!} a_k z^{-1}}{\Gamma(n + \alpha + 1) - \sum \frac{\Gamma(n+\alpha+k)}{(k-1)!} a_k z^{k-1}} \right\} > \alpha/(\alpha + 1)$$

for $z \in U$. Choose values of z on the real axis so that $D^{n+\alpha+1} f(z)/D^{n+\alpha} f(z)$ is real. Upon clearing the denominator in (3.3) and letting $z \rightarrow 1^-$ through real values, we can observe that

$$\begin{aligned} &\Gamma(n + \alpha + 1) - \sum \frac{\Gamma(n + \alpha + k + 1)}{(n + \alpha + 1)(k - 1)!} a_k \\ &\geq \frac{\alpha}{\alpha + 1} \left\{ \Gamma(n + \alpha + 1) - \sum \frac{\Gamma(n + \alpha + k)}{(k - 1)!} a_k \right\} \end{aligned}$$

which implies (3.1).

Finally we can show that the function $f(z)$ given by (3.2) is an extremal function for the theorem. This completes the proof of Theorem 1.

COROLLARY 1. *Let the function $f(z)$ defined by (1.7) be in the class $R(n + \alpha)$ for $n \in N \cup \{0\}$ and $0 \leq \alpha < 1$. Then*

$$a_k \leq \frac{(k-1)! \Gamma(n + \alpha + 2)}{(n + \alpha k + k) \Gamma(n + \alpha + k)}$$

for $k \geq 2$. The equality holds for the function $f(z)$ of the form (3.2).

Applying Theorem 1, we prove

THEOREM 2. *Let the function $f(z)$ defined by (1.7) be in the class $R(n + \alpha)$ for $n \in N \cup \{0\}$ and $0 \leq \alpha < 1$. Then*

$$|z| - (n + 2\alpha + 2)^{-1} |z|^2 \leq f(z) \leq |z| + (n + 2\alpha + 2)^{-1} |z|^2$$

for $z \in U$. The result is sharp.

Proof. Since $f(z)$ belongs to the class $R(n + \alpha)$, by using Theorem 1, we have

$$(n + 2\alpha + 2) \Gamma(n + \alpha + 2) \sum a_k \leq \sum \frac{(n + \alpha k + k) \Gamma(n + \alpha + k)}{(k-1)!} a_k \leq \Gamma(n + \alpha + 2)$$

which gives that $\sum a_k \leq (n + 2\alpha + 2)^{-1}$. Hence we can see that

$$\begin{aligned} |f(z)| &\geq |z| - |z|^2 \sum a_k \geq |z| - (n + 2\alpha + 2)^{-1} |z|^2, \\ |f(z)| &\leq |z| + |z|^2 \sum a_k \leq |z| + (n + 2\alpha + 2)^{-1} |z|^2 \end{aligned}$$

for $z \in U$.

Further, by taking the function

$$f(z) = z - (n + 2\alpha + 2)^{-1} z^2$$

we can prove that the result of the theorem is sharp.

COROLLARY 2. *Let the function $f(z)$ defined by (1.7) be in the class $R(n + \alpha)$ for $n \in N \cup \{0\}$ and $0 \leq \alpha < 1$. Then $f(z)$ is included in a disk with its center at the origin and radius R given by $R = (n + 2\alpha + 3)(n + 2\alpha + 2)^{-1}$.*

THEOREM 3. *Let the function $f(z)$ defined by (1.7) be in the class $R(n + \alpha)$ for $n \in N \cup \{0\}$ and $0 \leq \alpha < 1$. Then, for $z \in U$,*

$$1 - (2 + \alpha)|z|/(n + 2\alpha + 2) \leq |f'(z)| \leq 1 + (2 + \alpha)|z|/(n + 2\alpha + 2).$$

Proof. In view of Theorem 1, we have

$$\begin{aligned} &(n + 2\alpha + 2) \Gamma(n + \alpha + 2) (2 + \alpha)^{-1} \sum k a_k \\ &\leq \sum \frac{(n + \alpha k + k) \Gamma(n + \alpha + k)}{(k-1)!} a_k \leq \Gamma(n + \alpha + 2) \end{aligned}$$

which implies that

$$(3.4) \quad \sum ka_k \leq (2 + \alpha)(n + 2\alpha + 2)^{-1}.$$

Consequently, by using (3.4), we have

$$\begin{aligned} |f'(z)| &\geq 1 - |z| \sum ka_k \geq 1 - (2 + \alpha)|z|/(n + 2\alpha + 2), \\ |f'(z)| &\leq 1 + |z| \sum ka_k \leq 1 + (2 + \alpha)|z|/(n + 2\alpha + 2) \end{aligned}$$

for $z \in U$.

Remark. We have not been able to obtain a sharp estimate $|f'(z)|$ in Theorem 3.

COROLLARY 3. *Let the function $f(z)$ defined by (1.7) be in the class $R(n + \alpha)$ for $n \in N \cup \{0\}$ and $0 \leq \alpha < 1$. Then $f'(z)$ is included in a disk with its center at the origin and radius R' given by $R' = (n + 3\alpha + 4)(n + 2\alpha + 2)^{-1}$.*

The following sharp estimation for $|f'(z)|$ is due to Professor M. Obradović.

THEOREM 3'. *Let the function $f(z)$ defined by (1.7) be in the class $R(n + \alpha)$ for $n \in N \cup \{0\}$ and $0 \leq \alpha < 1$. Then, for $z \in U$,*

$$1 - 2|z|/(n + 2\alpha + 2) \leq |f'(z)| \leq 1 + 2|z|/(n + 2\alpha + 2).$$

The result is sharp.

Proof. By using Theorem 1, we have

$$\begin{aligned} \Gamma(n + \alpha + 2) &\leq \sum \frac{(n + \alpha k + k)\Gamma(n + \alpha + k)}{(k - 1)!} a_k \\ &= \sum \frac{(n + \alpha k + k)\Gamma(n + \alpha + k)}{k!} k a_k. \end{aligned} \quad (*)$$

Consider the expression

$$\frac{(n + \alpha k + k)\Gamma(n + \alpha + k)}{k!} = \frac{(n + \alpha k + k)(n + \alpha k + k - 1) \dots (n + \alpha + 2)\Gamma(n + \alpha + 2)}{k!}$$

for $n = 0$:

$$(\alpha k + k)\Gamma(\alpha + k)/k! \geq (\alpha + 1)\Gamma(\alpha + 2)$$

for $n \in N$:

$$(n + \alpha k + k)\Gamma(n + \alpha + k)/k! \geq (n + 2\alpha + 2)\Gamma(n + \alpha + 2)/2.$$

So, we can conclude that, for every $n \in N \cup \{0\}$, we have

$$(n + \alpha k + k)\Gamma(n + \alpha + k)/k! \geq (n + 2\alpha + 2)\Gamma(n + \alpha + 2)/2,$$

and then from (*) we obtain

$$\sum ka_k \geq 2/(n + 2\alpha + 2).$$

The rest of the proof is as for Theorem 3.

Furthermore, that this estimation is the best possible is shown by the function

$$f(z) = z - z^2/(n + 2\alpha + 2).$$

We derive some distortion inequalities for the fractional calculus of functions belonging to the class $R(n + \alpha)$.

THEOREM 4. *Let the function $f(z)$ defined by (1.7) be in the class $R(n + \alpha)$ for $n \in N \cup \{0\}$ and $0 \leq \alpha < 1$. Then*

$$(3.5) \quad |D_z^{-\lambda} f(z)| \geq \frac{|z|^{1+\lambda}}{\Gamma(2 + \lambda)} \left\{ 1 - \frac{2|z|}{(2 + \lambda)(n + 2\alpha + 2)} \right\}$$

$$(3.6) \quad |D_z^{-\lambda} f(z)| \leq \frac{|z|^{1+\lambda}}{\Gamma(2 + \lambda)} \left\{ 1 + \frac{2|z|}{(2 + \lambda)(n + 2\alpha + 2)} \right\}$$

for $\lambda > 0$ and $z \in U$. The results in (3.5) and (3.6) are sharp.

Proof. Let

$$(3.7) \quad F(z) = \Gamma(2 + \lambda)z^{-\lambda}D_z^{-\lambda}f(z) = z - \sum \frac{k!\Gamma(2 + \lambda)}{\Gamma(k + 1 + \lambda)}a_k z^k$$

for $\lambda > 0$. Then we can observe that

$$(3.8) \quad 0 < k!\Gamma(2 + \lambda)/\Gamma(k + 1 + \lambda) < 2/(2 + \lambda)$$

for $\lambda > 0$ and $k \leq 2$. Hence, with the help of (3.8) and Theorem 1, we have

$$|F(z)| \geq |z| - \frac{2}{2 + \lambda}|z|^2 \sum a_k \geq |z| - \frac{2|z|^2}{(2 + \lambda)(n + 2\alpha + 2)}$$

$$|F(z)| \leq |z| + \frac{2}{2 + \lambda}|z|^2 \sum a_k \leq |z| + \frac{2|z|^2}{(2 + \lambda)(n + 2\alpha + 2)}$$

which give (3.5) and (3.6), respectively. Further, we can see that the results in (3.5) and (3.6) are sharp for the function $f(z)$ defined by

$$f(z) = z - \frac{z^2}{n + 2\alpha + 2}.$$

COROLLARY 4. *Let the function $f(z)$ defined by (1.7) be in the class $R(n + \alpha)$ for $n \in N \cup \{0\}$ and $0 \leq \alpha < 1$. Then $D_z^{-\lambda} f(z)$ is included in a disk with its center at the origin and radius $R^{-\lambda}$ given by*

$$R^{-\lambda} = \frac{1}{\Gamma(2 + \lambda)} \left\{ 1 + \frac{2}{(2 + \lambda)(n + 2\alpha + 2)} \right\}.$$

where $\lambda > 0$.

Applying Theorem 3' to the function $F(z)$, we have

THEOREM 5. *Let the function $f(z)$ defined by (1.7) be in the class $R(n + \alpha)$ for $n \in N \cup \{0\}$ and $0 \leq \alpha < 1$. Then*

$$(3.9) \quad |D_z^{-\lambda} f(z)| \geq \text{Max} \left\{ 0, \frac{|z|}{\Gamma(2 + \lambda)} \left(1 - \lambda \frac{(2 + \lambda)|z|}{(n + 2\alpha + 2)} \right) \right\},$$

$$(3.10) \quad |D_z^{-\lambda} f(z)| \leq \frac{|z|^\lambda}{\Gamma(2 + \lambda)} \left(1 + \lambda + \frac{(2 + \lambda)|z|}{(n + 2\alpha + 2)} \right)$$

for $\lambda > 0$ and $z \in U$.

Proof. It is easy to see that the function $F(z)$ defined by (3.7) is also the class $R(n + \alpha)$. Hence, by means of Theorem 3' we get

$$1 - 2|z|/(n + 2\alpha + 2) \leq F'(z) \leq 1 + 2|z|/(n + 2\alpha + 2)$$

for $z \in U$ which gives (3.9) and (3.10).

COROLLARY 5. *Let the function $f(z)$ defined by (1.7) be in the class $R(n + \alpha)$ $n \in N \cup \{0\}$ and $0 \leq \alpha < 1$. Then $D_z^{1-\lambda} f(z)$ is included in a disk with its center at the origin and radius $R^{1-\lambda}$ given by*

$$R^{1-\lambda} = \frac{1}{\Gamma(2 + \lambda)} \left(1 + \lambda + \frac{2 + \lambda}{n + 2\alpha + 2} \right), \text{ where } \lambda > 0.$$

4. Closure theorems

THEOREM 6. *Let the functions*

$$(4.1) \quad f_i(z) = z \sum a_{k,i} z^k \quad (a_{k,i} \geq 0)$$

be in the class $R(n + \alpha)$ for $n \in N \cup \{0\}$, $0 \leq \alpha < 1$ and every $i = 1, 2, 3, \dots, m$. Then the function $h(z)$ defined by

$$h(z) = \sum_{i=1}^m c_i f_i(z) \quad (c_i \geq 0)$$

is also in the class $R(n + \alpha)$, where $\sum_{i=1}^m c_i = 1$.

Proof. By means of the definition of $h(z)$, we can write

$$h(z) = z - \sum \left(\sum_{i=1}^m c_i a_{k,i} \right) z^k.$$

Further, by virtue of Theorem 1, we have

$$\sum \frac{(n + \alpha k + k)\Gamma(n + \alpha + k)}{(k - 1)!} a_{k,i} \leq \Gamma(n + \alpha + 2)$$

for every $i = 1, 2, 3, \dots, m$. Hence we can see that

$$\begin{aligned} & \sum \frac{(n + \alpha k + k)\Gamma(n + \alpha + k)}{(k - 1)!} \left(\sum_{i=1}^m c_i a_{k,i} \right) \\ & \sum_{i=1}^m c_i \left\{ \sum \frac{(n + \alpha k + k)\Gamma(n + \alpha + k)}{(k - 1)!} a_{k,i} \right\} \\ & \leq \left(\sum_{i=1}^m c_i \right) \Gamma(n + \alpha + 2) = \Gamma(n + \alpha + 2) \end{aligned}$$

which implies that $h(z)$ is in the class $R(n + \alpha)$ with the aid of Theorem 1.

THEOREM 7. *Let $f_1(z) = z$ and*

$$(4.2) \quad f_k(z) = z - \frac{(k - 1)\Gamma(n + \alpha + k)}{(n + \alpha k + k)\Gamma(n + \alpha + k)} z^k \quad (k \in N - \{1\}).$$

Then $f(z)$ is in the class $R(n + \alpha)$ for $n \in N \cup \{0\}$ and $0 \leq \alpha < 1$ if and only if it can be expressed in the form

$$(4.3) \quad f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z),$$

where $\lambda_k \geq 0$ for $k \in N$ and $\sum_{k=1}^{\infty} \lambda_k = 1$.

Proof. Suppose that

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z) = z - \sum \frac{(k - 1)\Gamma(n + \alpha + 2)}{(n + \alpha k + k)\Gamma(n + \alpha + k)} \lambda_k z^k.$$

Then we obtain that

$$\sum \left\{ \frac{(n + \alpha k + k)\Gamma(n + \alpha + k)}{(k - 1)!} \frac{(k - 1)\Gamma(n + \alpha + 2)}{(n + \alpha k + k)\Gamma(n + \alpha + k)} \lambda_k \right\} \leq \Gamma(n + \alpha + 2).$$

This shows that $f(z)$ belongs to the class $R(n + \alpha)$ for $n \in N \cup \{0\}$ and $0 \leq \alpha < 1$. Again, by using Theorem 1.

For the converse, suppose that $f(z)$ belongs to the class $R(n + \alpha)$ for $n \in N \cup \{0\}$ and $0 \leq \alpha < 1$. Again, by using Theorem 1, we can observe that

$$a_k \leq \frac{(k - 1)\Gamma(n + \alpha + 2)}{(n + \alpha k + k)\Gamma(n + \alpha + k)} \quad (k \in N - \{1\}).$$

Now, setting

$$\lambda_k = \frac{(n + \alpha + k)\Gamma(n + \alpha + k)}{(k - 1)\Gamma(n + \alpha + 2)} a_k \quad (k \in N - \{1\})$$

and $\lambda_1 = 1 - \sum \lambda_k$, we have the representation (4.3). This completes the proof of Theorem 7.

5. Modified Hadamard product

Let $f(z)$ be defined by (1.7) and $g(z)$ be defined by

$$g(z) = z - \sum b_k z^k \quad (b_k \geq 0).$$

Further, let $f * g(z)$ denote the modified Hadamard product of $f(z)$ and $g(z)$, that is, $f * g(z) = z - \sum a_k b_k z^k$.

THEOREM 8. *Let the functions $f_i(z)$ defined by (4.1) be in the classes $R(n_i + \alpha_i)$ for $n_i \in N \cup \{0\}$, $0 \leq \alpha < 1$ and each $i = 1, 2, 3, \dots, m$, respectively. Then the modified Hadamard product $f_1 * f_2 * \dots * f_m(z)$ belongs to the class $R(n + \alpha)$, where $n + \alpha = \text{underset } 1 \leq i \leq m \rightarrow \text{Min}\{n_i + \alpha_i\}$.*

Proof. Since $f_i(z) \in R(n + \alpha_i)$ for each $i = 1, 2, 3, \dots, m$, in view of Theorem 1, we get

$$(5.1) \quad a_{k,i} \leq 1/(n_i + 2\alpha + 2) \quad (i = 1, 2, 3, \dots, m).$$

Therefore we can show that

$$\begin{aligned} & \sum \frac{(n + \alpha k + k)\Gamma(n + \alpha + k)}{(k - 1)!} \left(\prod_{i=1}^m a_{k,i} \right) \\ & \leq \frac{(n + 2\alpha + k)\Gamma(n + \alpha + k)}{\prod_{i=1}^m (n_i + 2\alpha_i + 2)} \leq \Gamma(n + \alpha + 2) \end{aligned}$$

with the help of (5.1) and Theorem 1. Thus we have Theorem 8.

COROLLARY 6. *Let the functions $f_i(z)$ defined by (4.1) be in the same class $R(n + \alpha)$ for $n \in N \cup \{0\}$, $0 \leq \alpha < 1$ and every $i = 1, 2, 3, \dots, m$. Then the modified Hadamard product $f_1 * f_2 * \dots * f_m(z)$ also belongs to the class $R(n + \alpha)$.*

6. Radii of starlikeness and convexity

We determine the radii of starlikeness and convexity of functions $f(z)$ belonging to the class $R(n + \alpha)$.

THEOREM 9. *Let the function $f(z)$ defined by (1.7) be in the class $R(n + \alpha)$ for $n \in N \cup \{0\}$ and $0 \leq \alpha < 1$. Then $f(z)$ is starlike of order β ($0 \leq \beta < 1$) in the disk $|z| < r_0$, where*

$$r_0 = \inf_{k \in N - \{1\}} \left\{ \frac{(1 - \beta)(n + \alpha k + k)\Gamma(n + \alpha + k)}{(k - \beta)(k - 1)\Gamma(n + \alpha + 2)} \right\}^{1/(k-1)}$$

with equality for the function $f(z)$ given by (4.2).

Proof. It suffices to show that

$$|zf'(z)/f(z) - 1| < 1 - \beta$$

for $|z| < r_0$. Now, we can observe that

$$|zf'(z)/f(z) - 1| \leq \frac{\sum (k-1)a_k |z|^{k-1}}{1 - \sum a_k |z|^{k-1}} \leq 1 - \beta$$

if and only if

$$\sum ((k-\beta)/(1-\beta))a_k |z|^{k-1} \leq 1.$$

Hence, by virtue of Theorem 1, we need only find values of $|z|$ for which

$$\left(\frac{k-\beta}{1-\beta}\right) |z|^{k-1} \leq \frac{(n+\alpha k+k)\Gamma(n+\alpha+k)}{(k-1)!\Gamma(n+\alpha+2)} \quad (k \geq 2),$$

which will be true when $|z| \leq r_0$. This completes the proof of the Theorem.

Finally, we have

THEOREM 10. *Let the function $f(z)$ defined by (1.7) be in the class $R(n+\alpha)$ for $n \in N \cup \{0\}$ and $0 \leq \alpha < 1$. Then $f(z)$ is convex of order β ($0 \leq \beta < 1$) in the disk $|z| < r_1$, where*

$$r_1 = \inf_{k \in N - \{1\}} \left\{ \frac{(1-\beta)(n+\alpha k+k)\Gamma(n+\alpha+k)}{k!\Gamma(n+\alpha+2)} \right\}^{1/(k-1)}.$$

Proof. Since $f(z)$ defined by (1.7) is convex of order β if and only if $zf'(z)$ is starlike of order β , we can show that the proof of the Theorem follows the proof of Theorem 9, with a_k replaced by ka_k .

COROLLARY 7. *Let the function $f(z)$ defined by (1.7) be in the class $R(n+\alpha)$ for $n \in N \cup \{0\}$ and $0 \leq \alpha < 1$. Then $f(z)$ is univalent and starlike for $|z| < r_2$, where*

$$r_2 = \inf_{k \in N - \{1\}} \left\{ \frac{(n+\alpha k+k)\Gamma(n+\alpha+k)}{k!\Gamma(n+\alpha+2)} \right\}^{1/(k-1)}.$$

Proof. By taking $\beta = 0$ in Theorem 9, we have the Corollary.

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Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577
Japan

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