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QUASI-RADICALS AND RADICALS IN CATEGORIES

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Abstract. In a category $\mathcal{K}$, if $\mathcal{E}$ is a class of epimorphisms and $\mathcal{M}$ a class of monomorphisms, a funtion J_r called an $(\mathcal{E}, \overline{\mathcal{M}})$-quasi-radical, is defined which assigns to an object an $\mathcal{M}$-sink and a function J_c, called an $(\overline{\mathcal{E}}, \mathcal{M})$-quasi-coradical, is defined which assigns to an object an $\mathcal{E}$-source. With J_r are associated two object classes $\mathbf{R}_r$ and $\mathbf{S}_r$ called the quasi-radical class and the quasi-semisimple class respectively. With J_c are associated two object classes $\mathbf{R}_c$ and $\mathbf{S}_c$, called the quasi-coradical class and the quasi-cosemisimple class respectively. Using these notions, an $(\mathcal{E}, \overline{\mathcal{M}})$-radical is a pair (J_r, J_c) where J_r, is a quasiradical, J_c a quasi-coradical and for which $\mathbf{R}_r = \mathbf{R}_c$ and $\mathbf{S}_r = \mathbf{S}_c$. Among others it is shown that $\mathbf{R}_r = \mathbf{R}_c$ is a radical class and $\mathbf{S}_r = \mathbf{S}_c$ is a semisimple class.

Introduction. Radical theory in categories have been studied since the early sixties by, among others, Sulgeifer [**12**], Rjabuhin [**10**], Sulinski [**13**], Wiegandt [**16**], Carreau [**4**] and Holcombe and Walker [**7**]. Conditions were imposed on the category to make it very "ring-like". For the characterization of radical and semisimple classes in categories, these conditions were essential. The middle of the previous decade saw the development of a radical theory in various branches of mathematics outside the traditional algebraic structures (rings, groups, modules). Arhangelskii and Wiegandt [**2**] has shown that the connectedness theory of topological spaces corresponds to the radical theory of algebraic structures. A connectedness theory for graphs has been developed by Fried and Wiegandt [**5**] and for S-acts by Amin, Lex and Wiegandt in [**1**] and [**8**]. These developments has made the above mentioned "ring-like" categories inadequate for the study of a general radical theory (to include the connectedness theory) in such categories. This situation was partly remedied in [**15**] where radical classes were defined in a category that could be algebraic (groups, rings or modules), topological or a category of graphs. Once again, various conditions had to be imposed on the category to obtain significant results as far as characterizations are concerned. Nevertheless, it included all the above mentioned categories except the category of S-acts. Obviously one would like as few conditions as possible. In [**3**] radical classes were defined in a category without any restrictions on it. Under these circumstances one could not wish to

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get far, let alone distinguishing the radical subobject of an object or the maximal semisimple image of an object. Our aim in this paper is to start at the other end. We begin with the radical subobject (or subobjects) and the maximal semisimple image and use these to obtain a radical class. This gives rise to the definition of an $(\mathcal{E},\overline{\mathcal{M}})$-quasi-radical and an $(\overline{\mathcal{E}},\mathcal{M})$-quasi-coradical in section 1 where $\mathcal{E}$ is a class of epimorphisms and $\mathcal{M}$ a class of monomorphisms. These concepts are shown to have functorial properties and their relationship with coreflective and reflective subcategories are also given. Combining the above two concepts, we define an $(\mathcal{E},\overline{\mathcal{M}})$-radical in section 2 and it is shown to have the desired properties. Lastly in section 3 we discuss the $(\mathcal{E},\overline{\mathcal{M}})$-radical in a category which satisfies the usual conditions. Its correspondence with radical classes and radical functors are given.

1. $(\mathcal{E},\overline{\mathcal{M}})$-quasiradical and $(\overline{\mathcal{E}},\mathcal{M})$-quasi-coradicals.

Let $\mathcal{K}$ be any category. By $\mathrm{Ob}\,\mathcal{K}$, $\mathrm{Mono}\,\mathcal{K}$, $\mathrm{Epi}\,\mathcal{K}$, $\mathrm{Iso}\,\mathcal{K}$ and $\mathrm{Id}\,\mathcal{K}$ we'll denote the class of all objects, monomorphisms, epimorphsms, isomorphisms and identity morphisms of $\mathcal{K}$ respectively. As usual, for a morphism $\alpha : A \to B$, $\mathrm{dom}\,\alpha = A$ and $\mathrm{cod}\,\alpha = B$.

Let $\mathcal{E} \subseteq \mathrm{Epi}\,K$ and $\mathcal{M} \subseteq \mathrm{Mono}\,\mathcal{K}$ with $\mathrm{Id}\,K subseteq\mathcal{E}\cap\Updownarrow$ and such that both $\mathcal{E}$ and $\mathcal{M}$ are closed with respect to composition with isomorphisms.

An $\mathcal{M}$*-sink of an object* A, is a sink of the form $(\alpha_i : A_i \to A)_I$ with $\alpha_i \in \mathcal{M}$ for all $i \in I$ and $|I| \geq 1$. Such a sink is said to be *totally unordered* if for any i and j in I, if $(A_i,\alpha_i) \leq (A_j,\alpha_j)$ then $(A_i,\alpha_i) = (A_j,\alpha_j)$. By $\alpha \in (\alpha_i : A_i \to A)_I$ we'll mean there is an $i_0 \in I$ such that $\alpha = \alpha_{i_0} : A_{i_0} \to A$. Let $\mathcal{M}_A$ be the class of all the totally unordered sinks of the object A and let $\overline{\mathcal{M}} = \bigcup\{\mathcal{M}_A |\, A \in \mathrm{Ob}\,K\}$. The dual notions are $\mathcal{E}$-source of an object, $\mathcal{E}_A$ and $\overline{\mathcal{E}}$. We have to mention that by dualising we only interchange the classes $\mathcal{E}$ and $\mathcal{M}$ and change the direction of arrows although we do not require an epimorphism in $\mathcal{E}$ to have the dual properties of a monomorphism from $CalM$.

1.1 *Definition.* An $(\mathcal{E},\overline{\mathcal{M}})$-quasi-radical J_r in $\mathcal{K}$ is a function $J_r : \mathrm{Ob}\,\mathcal{K} \to \overline{\mathcal{M}}$ which assigns to each object A an $\mathcal{M}$-sink from $\mathcal{M}_A$, say $\alpha_i : A_i \to A)_I$, such that the following two conditions hold:

(i) For any $\varphi : A \to B$, $\varphi \in \mathcal{M}$ and $\alpha_i \in J_rA$, there exists a $\beta_j \in J_rB$ and a morphism φ_r such that $\varphi_r\beta_j = \alpha_i\varphi$.

(ii) For each $A \in \mathrm{Ob}\,\mathcal{K}$ and $\alpha_i \in J_rA$, $\gamma \in \mathcal{E}\cap\mathcal{M}$ must hold for all $\gamma \in J_r(\mathrm{dom}\,\alpha_i)$.

If J_r is an $(\overline{\mathcal{E}},\mathcal{M})$-quasi-radical such that for every $A \in \mathrm{Ob}\,\mathcal{K}$, the sink J_rA consists of exactly one morphism, we will distinguish it from the usual by saying J_r is an $(\mathcal{E},\mathcal{M})$*-quasi-radical.*

Note that in condition (i) above, φ_r is a monomorphism and for any $\varphi \in M$ and $\alpha_i \in J_rA$, once $\beta_j \in J_rB$ has been fixed, φ_r is uniquely determined by the commutative property. Let us fix some notations. For any object A, $\alpha_r : A_r \to A$ will always represent a morphism from the sink J_rA and I_A will always be the index set of the sink J_rA.

1.1* *Definition.* An $(\mathcal{E}, \overline{\mathcal{M}})$-quasi-coradical J_c in $\mathcal{K}$ is a function $J_c : \mathrm{Ob}\,\mathcal{K} \to \mathcal{E}$ which assigns to each object A an $\mathcal{E}$-source from $\mathcal{E}_A$, say $(\alpha_i^* : A \to A_i^*)I_A^*$, such that the following two conditions hold:

(i) For any $\varphi : A \to B$, $\varphi \in \mathcal{E}$, and $\alpha_i^* \in J_cA$, there exists a $\beta_j^* \in J_cB$ and a morphism φ_c such that $\alpha_I^*\varphi_c = \varphi\beta_j^*$.

(ii) For each $A \subset \mathrm{Ob}\,\mathcal{K}$ and $\alpha_i^* \in J_cA$, $\gamma \in \mathcal{E} \cap \mathcal{M}$ must hold for all $\gamma \in J_c(\mathrm{cod}\,\alpha_i^*)$.

Once again, if the source J_cA consists of exactly one morphism for each $A \in \mathrm{Ob}\,\mathcal{K}$, then we will call J_c an $(\mathcal{E}, \mathcal{M})$-*quasi-coradical.* For any object A, $\alpha_c : A \to A_c$ will represent a morphism from the source J_cA and I_A^* will always be the index set of the source J_cA.

1.2 *Examples.* Let $\mathcal{K}$ be the category *Rng* of all associative rings. Let $\mathcal{E}$ be the class of normal epimorphisms and $\mathcal{M}$ the class of normal monomorphisms. For any radical class $\mathbf{R}$ in $\mathcal{K}$, let $\mathbf{R}(A)$ denote the $\mathbf{R}$-radical of the ring A with $i_A : \mathbf{R}(A) \to A$ the inclusion and $\pi_A : A \to A/\mathbf{R}(A)$ the quotient. Then J_r and J_c, defined by $J_rA = i_A$ and $J_cA = \pi_A$, determines an $(\mathrm{Iso}\,\mathcal{K}, \mathcal{M})$-quasi-radical and an $(\mathcal{E}, \mathrm{Iso}\,\mathcal{K})$-quasi-coradical respectively.

2. If $\mathcal{K}$ is the category of groups, *Grp*, abelian groups *Ab*, R-modules R-mod or the category of alternative rings *Arng* with $\mathcal{E}$ the normal epimorphisrns and $\mathcal{M}$ the class of normal monorphisms, then any radical class determines an $(\mathrm{Iso}\,\mathcal{K}, \mathcal{M})$-quasi-radical and $(\mathcal{E}, \mathrm{Iso}\,\mathcal{K})$-quasi-coradical as in 1 above.

3. Let $\mathcal{K}$ be the category *Top* of topological spaces with $\mathcal{E}$ the class of all epimorphisms and $\mathcal{M}$ the class of all extremal monorphisms. Let $\mathbf{C}$ be any connectedness with corresponding disconnectednes $\mathbf{D}$ (see [**2**]). For each topological space X, let $f_X : X \to Y_X$ be the maximal $\mathbf{D}$-image of X with $(i_y : f_X^{-1}(y) \to X)_{Y_X}$ the sink of all the fibers of f_X. Then J_r and J_c, defined by $J_rX = (i_y : f_X^{-1}(y) \to X)_{Y_X}$ and $J_cX = f_X$, determines an $(\mathrm{Iso}\,\mathcal{K}, \overline{\mathcal{M}})$-quasi-radical and an $(\mathcal{E}, \mathrm{Iso}\,\mathcal{K})$-quasi-coradical respectively.

4. Let $\mathcal{K}$ be the category *Graph* of all undirected graphs which admit loops. Let $\mathcal{E}$ be the class of all epimorphisms and $\mathcal{M}$ the class of all extremal monorphisms. Then any connectedness (see [**5**]) determines an $(\mathrm{Iso}\,\mathcal{K}, \overline{\mathcal{M}})$-quasi-radical and an $(\mathcal{E}, \mathrm{Iso}\,\mathcal{K})$-quasi-coradical as in 3 above.

5. Let $\mathcal{K}$ be the category of integral domains (with identity) and morphisms the injective homomorphisms. Let $\mathcal{E} = \mathrm{Epi}\,\mathcal{K}$ and $\mathcal{M} = \mathrm{Id}\,\mathcal{K}$. For every integral domain A, let J_cA be the embedding of A into its field of quotients. Then J_c is an $(\mathcal{E}, \mathcal{M})$-quasi-coradical.

6. Let $\mathcal{K}$ be the category of Ω-groups. Let $\mathcal{E}$ be the class of surjective homomorphisms and $\mathcal{M}$ the class of injective homomorphisms. Let $\{P_\alpha |\, \alpha \in I\}$ be a set of n-ary relations (n a natural number greater than 1) defined on every $A \in \mathrm{Ob}\,\mathcal{K}$. We assume the following holds:

For any homomorphism $\varphi : A \to B$, if $(a_1, a_2, \ldots, a_n) \in P_\alpha$ in A, then $(\varphi a_1, \varphi a_2, \ldots, \varphi a_n) \in P_\alpha$ in B for any P_α.

If $A \in \mathrm{Ob}\,\mathcal{K}$ and $S \subseteq A$, then S is said to be closed under P_α if $y_1, y_2, \ldots, y_{n-1} \in S$ and $(y_1, y_2, \ldots, y_{n-1}, x) \in P_\alpha$ implies $x \in S$.

Let $A' = \bigcap\{B|\, B$ is an ideal in A and is closed under P_α for each $\alpha \in I\}$. Let J_rA be the inclusion of A' in A and J_cA the quotient. Then J_r is an $(\mathcal{E}, \mathcal{M})$-quasi-radical and J_c is an $(\mathcal{E}, \mathcal{M})$-quasi-co-radical.

7. Let $\mathcal{K}$ be a category of S-acts (cf. [**8**]) with $\mathcal{E}$ the class of Rees factor mappings and $\mathcal{M}$ the embeddings of subacts. Then any connectedness determines and $(\mathrm{Iso}\,\mathcal{K}, \overline{\mathcal{M}})$-quasi-radical and an $(\mathcal{E}, \mathrm{Iso}\,\mathcal{K})$-quasi-coradical as in example 3 above.

An abundence of examples of $(\mathcal{E}, \mathcal{M})$-quasi-radical sand $(\mathcal{E}, \mathcal{M})$-quasi-coradicals are given by the next proposltion and its dual (cf. [**6**] §26). Firstly, we recall, an *$\mathcal{E}$-reflective subcategory* $\mathcal{A}$ of $\mathcal{K}$ is a subcategory $\mathcal{A}$ such that for each object A in $\mathcal{K}$ there exists a morphism $\pi_A : A \to A_A$ with $\pi_a \in \mathcal{E}$ and $A \in \mathrm{Ob}\,\mathcal{A}$ such that for any morphism $\beta : A \to A'$ with $A' \in \mathrm{Ob}\,\mathcal{A}$, there is a unique morphism $\gamma : A \to A'$ for which $\pi_A\gamma = \beta$ holds. The dual notion is *$\mathcal{M}$-coreflective subcategory*.

Proposition. *Let $\mathcal{A}$ be an $\mathcal{E}$-reflective subcategory of $\mathcal{K}$. Then $\mathcal{A}$ determine an $\mathcal{E}, \mathrm{Iso}\,\mathcal{K})$-quasi-coradical J_c in $\mathcal{K}$ where, for each $A \in \mathrm{Ob}\,\mathcal{K}$, J_cA is given by the $\mathcal{E}$-reflection of A in A.*

Proof. For each A if $J_cA = \alpha_c : A \to A_c$, then by definition α_c is the $\mathcal{E}$-reflection of A in $\mathcal{A}$. Let $\varphi : A \to B$ with $\varphi \in \mathcal{E}$. Using the properties of the reflection α_c, there exists a unique morphism φ_c such that $\alpha_c \cdot \varphi_c = \varphi\beta_c$ where $J_cB = \beta_c$. Lastly, for any object A in $\mathcal{K}$, the $\mathcal{E}$-reflection of A_c in $\mathcal{A}$ is an isomorphism.

1.3* Proposition. *Let $\mathcal{A}$ be an $\mathcal{M}$-coreflective subcategory of $\mathcal{K}$. Then $\mathcal{A}$ determines an $(\mathrm{Iso}\,\mathcal{K}, \mathcal{M})$-quasi-radical J_r in $\mathcal{K}$ where, for each $A \in \mathrm{Ob}\,\mathcal{K}$, J_rA is given by the $\mathcal{M}$-coreflection of A in $\mathcal{A}$.*

Before we give a partial convers to the above, we might just mention that any class of morfisms $\mathcal{P}$ with $\mathrm{Id}\,\mathcal{K} \subseteq \mathcal{P}$ and which is closed under composition, gives rise to a subcategory $\mathcal{P}(\mathcal{K})$ of $\mathcal{K}$ in the following way: $\mathrm{Ob}\,(\mathcal{P}(\mathcal{K})) = \mathrm{Ob}\,\mathcal{K}$ and the morphisms in $\mathcal{P}(\mathcal{K})$ are only those morphisms of $\mathcal{K}$ which are in $\mathcal{P}$.

1.4 Propositon. *Suppose $\mathcal{E}$ is closed under composition and let J_c be an $(\mathcal{E}, \mathrm{Iso}\,\mathcal{K})$-quasi-coradical in $\mathcal{K}$. Let $\mathcal{A}$ be the following subcategory of $\mathcal{E}(\mathcal{K})$: $\mathrm{Ob}\,\mathcal{A} = \{A \in \mathrm{Ob}\,\mathcal{K}|\, J_cA \in \mathrm{Iso}\,\mathcal{K}\}$ and $\mathrm{Hom}_{\mathcal{A}}(A, B) = \mathrm{Hom}_{\mathcal{E}(\mathcal{K})}(A, B)$, for all $A, B \in \mathrm{Ob}\,\mathcal{A}$. Then $\mathcal{A}$ is an $\mathcal{E}$-reflective subcategory of$\mathcal{E}(\mathcal{K})$.*

Proof. Obviously, for any $A \in \mathrm{Ob}\,\mathcal{E}(\mathcal{K})$, the $\mathcal{E}$-reflection is given by J_cA.

1.4* Proposition. *Suppose $\mathcal{M}$ is closed composition and let J_r be an $(\mathrm{Iso}\,\mathcal{K}, \mathcal{M})$-quasi-radical in $\mathcal{K}$. Let $\mathcal{A}$ be the following subkategory of $\mathcal{M}(\mathcal{K})$: $\mathrm{Ob}\,\mathcal{A} = \{A \in \mathrm{Ob}\,\mathcal{K}|\, J_rA \in \mathrm{Iso}\,\mathcal{K}\}$ and $\mathrm{Hom}_{\mathcal{A}}(A, B) = \mathrm{Hom}_{\mathcal{M}(\mathcal{K})}(A, B)$ for all $A, B \in \mathrm{Ob}\,\mathcal{A}$. Then $\mathcal{A}$ is an $\mathcal{M}$-coreflective subcategory of $\mathcal{M}(\mathcal{K})$.*

$(\mathcal{E},\mathcal{M})$-quasi-radicals and $(\mathcal{E},\mathcal{M})$-quasi-coradicals have functorial properties which are given by the following:

1.5 PROPOSITION. *Suppose $\mathcal{E}$ is closed under composition and let J_c be an $(\mathcal{E},\mathcal{M})$-quasi-coradical in $\mathcal{K}$. Then*

(i) $F : \mathcal{E}(\mathcal{K}) \to \operatorname{Epi}\mathcal{K}$ defined by $FA = A_c$ and $F_\varphi = \varphi_c$, is a functor.

(ii) $\eta : \operatorname{Ob}\mathcal{E}(\mathcal{K}) \to \mathcal{E}(\mathcal{K})$ defined by $\eta A = J_c A$ for $A \in \operatorname{Ob}\mathcal{E}(\mathcal{K})$ is a natural transformation from $1_{\mathcal{E}(\mathcal{K})}$ to F,

(iii) *$\eta FA \in \mathcal{E} \cap \mathcal{M}$ for all $A \in \operatorname{Ob}\mathcal{E}(\mathcal{K})$.*

If $\mathcal{M} = \operatorname{Id}\mathcal{K}$ and $\varphi_c \in \mathcal{E}$ for all $\varphi \in \mathcal{E}$, the following also hold:

(iv) *$\eta FA = 1_{FA}$ for all $A \in \operatorname{Ob}\mathcal{E}(\mathcal{K})$.*

(v) $F^2 = F$ (i.e. *$F(FA) = FA$ for all $A \in \operatorname{Ob}\mathcal{E}(\mathcal{K})$*).

1.5* PROPOSITION. *Suppose $\mathcal{M}$ is closed under composition and let J_r be an $(\mathcal{E},\mathcal{M})$-quasi-radical in $\mathcal{K}$. Then*

(i) *$F : \mathcal{M}(\mathcal{K}) \to \operatorname{Mono}\mathcal{K}$ defined by $FA = A_r$ and $F\varphi = \varphi_r$ is a functor.*

(ii) *$\eta : \operatorname{Ob}\mathcal{M}(\mathcal{K}) \to \mathcal{M}(\mathcal{K})$ defined by $\eta A = J_r A$ for $A \in \operatorname{Ob}\mathcal{M}(\mathcal{K})$ is a natural transformation from F to $1_{\mathcal{M}(\mathcal{K})}$.*

(iii) *$\eta FA \in \mathcal{E} \cap \mathcal{M}$ for all $A \in \operatorname{Ob}\mathcal{M}(\mathcal{K})$.*

If $\mathcal{E} = \operatorname{Id}\mathcal{K}$ and $\varphi_r \in \mathcal{M}$ for all $\varphi \in \mathcal{M}$, the following also hold:

(iv) *$\eta FA = 1_{FA}$ for all $A \in \operatorname{Ob}\mathcal{M}(\mathcal{K})$.*

(v) *$F^2 = F$.*

The converse of the above is given by the next two propositions. As in the above, the proofs follow directly from the definitions and are omitted.

1.6 PROPOSITION. *Suppose $\mathcal{E}$ is closed under composition. Let $F : \mathcal{E}(\mathcal{K}) \to \operatorname{Epi}\mathcal{K}$ be a functor with $\eta : \operatorname{Ob}\mathcal{E}(\mathcal{K}) \to \mathcal{E}(\mathcal{K})$ a natural transformation from $1_{\mathcal{E}(\mathcal{K})}$ to F such that $\eta FA \in \mathcal{E} \cap \mathcal{M}$ for all $A \in \operatorname{Ob}\mathcal{E}(\mathcal{K})$. Then J_r defined by $J_c A = \eta A$ is an $(\mathcal{E},\mathcal{M})$-quasi-coradical in $\mathcal{K}$.*

1.6* PROPOSITION. *Suppose $\mathcal{M}$ is closed under composition. Let $F : \mathcal{M}(\mathcal{K}) \to \operatorname{Mono}\mathcal{K}$ be a functor with $\eta : \operatorname{Ob}\mathcal{M}(\mathcal{K}) \to \mathcal{M}(\mathcal{K})$ a natural transformation from F to $1_{\mathcal{M}(\mathcal{K})}$ such that $\eta FA \in \mathcal{E} \cap \mathcal{M}$ for all $A \in \operatorname{Ob}\mathcal{M}(\mathcal{K})$. Then J_r defined by $J_r A = \eta A$ is an $(\mathcal{E},\mathcal{M})$-quasi-radical in $\mathcal{K}$.*

2. The $(\mathcal{E},\overline{\mathcal{M}})$-radical.

Let J_r be an $(\mathcal{E},\overline{\mathcal{K}})$-quasi-radical in $\mathcal{K}$. Then the classes $\mathbf{R}_r = \{A \in \operatorname{Ob}\mathcal{K} | \, \alpha_r \in \operatorname{Iso}\mathcal{K}$ for all $\alpha_r \in J_r A\}$ and $S_r = \{A \in \operatorname{Ob}\mathcal{K} | \, \alpha_r$ is constant for all $\alpha_r \in J_r A\}$ are called the *quasi radical class* and *quasi-semisimple class* of J_r respectively. Likewise, if J_c is an $(\mathcal{E},\mathcal{M})$-quasi-coradical in $\mathcal{K}$ the clasess $\mathbf{R}_c = \{A \in \operatorname{Ob}\mathcal{K} | \, \alpha_c \in \operatorname{Iso}\mathcal{K}$ for all $\alpha_c \in J_c A\}$ are called the *quasi-coradical class* and *quasi-cosemisimle class* off J_c respectively.

We need the following notions from [**3**]:

1. A class of objects $\mathbf{T}$ in $\mathcal{K}$ is called the class of *trivial objects* if it satisfies the following three conditions:

(T1) If there is a constant epimorphism $A \to B$, then $B \in \mathbf{T}$.

(T2) If there is a constant monomorphism $C \to D$, then $C \in \mathbf{T}$.

(T3) If $T \in \mathbf{T}$, then every morphism $A \to T$ and $T \to B$ is constant.

We will suppose that the class $\mathbf{T}$ exists in $\mathcal{K}$. Then $\mathbf{S}_r$ and $\mathbf{R}_c$ can be rewritten as $\mathbf{S}_r = \{A \in \operatorname{Ob}\mathcal{K} | \, A_r \in \mathbf{T}$ for all $\alpha_r \in J_r A\}$ and $\mathbf{R}_c = \{A \in \operatorname{Ob}\mathcal{K} | \, A_c \in \mathbf{T}$ for all $\alpha_c \in J_c A\}$. It is then easy to see that $\mathbf{S}_r$ is $\mathcal{M}$-hereditary and that $\mathbf{R}_c$ is $\mathcal{E}$-cohereditary.

2. $\mathbf{A} \subseteq \operatorname{Ob}\mathcal{K}$ is an $(\mathcal{E}, \mathcal{M})$-*radical class* if the following condition is satisfied: $A \in \mathbf{A}$ if and only if for any $\varphi : A \to B$, $\varphi \in E$ and $B \notin T$, there exists a $\psi : I \to B$ with $\psi \in \mathcal{M}$, $I \notin \mathbf{T}$ and $I \in \mathbf{A}$.

3. $\mathbf{A} \subseteq \operatorname{Ob}\mathcal{K}$ is an $(\mathcal{E}, \mathcal{M})$-*semisimple class* if the following condition is satisfied: $A \in \mathbf{A}$ if and only if for every $\varphi : A \to B$, $\varphi \in M$ and $B \notin T$, there exists a $\psi : B \to I$ with $\psi \in \mathcal{E}$, $I \notin \mathbf{T}$ and $I \in \mathbf{A}$.

4. The operators $\mathcal{R}_{\mathcal{E}}$ and $\delta_{\mathcal{M}}$ on a class of objects $\mathbf{A}$ are given by: $\mathcal{R}_{\mathcal{E}}\mathbf{A} = \{A \in \operatorname{Ob}\mathcal{K} | \, B \notin \mathbf{A}$ for every $\varphi : B \to A$ with $\varphi \in M$ and $B \notin \mathbf{T}\}$ and $\delta_{\mathcal{M}}\mathbf{A} = \{A \in \operatorname{Ob}\mathcal{K} | \, B \notin \mathbf{A}$ for every $\varphi : A \to B$ with $\varphi \in M$ and $B \notin \mathbf{T}\}$. Then $\mathbf{A}$ is an $(\mathcal{E}, \mathcal{M})$-radical class if and only if $\mathbf{A} = \mathcal{R}_{\mathcal{E}}\delta_{\mathcal{M}}\mathbf{A}$ and if $\mathbf{A}$ is an $(\mathcal{E}, \mathcal{M})$-radical class, then $\mathcal{S}_{\mathcal{M}}\mathbf{A}$ is an $(\mathcal{E}, \mathcal{M})$-semisimple class. Likewise, $\mathbf{B}$ is an $(\mathcal{E}, \mathcal{M})$-semisimple class if and only if $\mathbf{B} = \delta_{\mathcal{M}}\mathcal{R}_{\mathcal{E}}\mathbf{B}$ and if $\mathbf{B}$ is an $(\mathcal{E}, \mathcal{M})$-semisimple class, then $\mathcal{R}_{\mathcal{E}}\mathbf{B}$ is an $(\mathcal{E}, \mathcal{M})$-radical class.

A natural condition associated with a reflective (or coreflective) subcategory $\mathcal{A}$ is that $\operatorname{Ob}\mathcal{A}$ (must be an isomorphism closed class.I.e., iF $A \in \operatorname{Ob}\mathcal{A}$ and if $\xi : A \to B$ is an isomorphism, then $B \in \operatorname{Ob}\mathcal{A}$ must hold. We will henceforth assume that the above condition holds for any reflective or coreflective subcategory.

2.1 Proposition. *Let $\mathcal{A}$ be an $\mathcal{E}$-reflective subcategory of $\mathcal{K}$. Then $\mathcal{A}$ determines an $(\mathcal{E}, \operatorname{Iso}\mathcal{K})$-quasi-coradical J_c in $\mathcal{K}$ with $S_c = \operatorname{Ob}\mathcal{A}$ and $\mathbf{R}_c = \mathcal{R}_{\mathcal{E}}(\operatorname{Ob}\mathcal{A})$.*

Proof. In view of Proposition 1.3 $J_c A = \alpha_c : A \to A_c$ is an $(\mathcal{E}, \operatorname{Iso}\mathcal{K})$-quasi-coradical where α_c is the $\mathcal{E}$-reflection of A in $\mathcal{A}$. We now show $\mathbf{S}_c = \operatorname{Ob}\mathcal{A}$. Let $A \in \mathbf{S}_c$. Then $J_c A = \alpha_c$ is an isomorphism. Hence $A \in \operatorname{Ob}\mathcal{A}$ follows.

Conversely, if $A \in \operatorname{Ob}\mathcal{A}$ then the reflection of A in $\mathcal{A}$, and hence $J_c A$, is an isomorphism. Lastly we show $\mathbf{R}_c = \mathcal{R}_{\mathcal{E}}(\operatorname{Ob}\mathcal{A})$. Let $A \in \mathbf{R}_c$. Then $J_c A$ is constant, i.e. $A_c \in \mathbf{T}$. Let $\varphi : A \to B$ and $B \in \operatorname{Ob}\mathcal{A}$. By definition of a reflection, there exists a unique $\varphi' : A_c \to B$ such that $\alpha_c \varphi' = \varphi$. From this it follows that φ is a constant epimorphism, i.e. $B \in \mathbf{T}$. Conversely, if $A \in \mathcal{R}_{\mathcal{E}}(\operatorname{Ob}\mathcal{A})$, then, because $\alpha_c = J_c A \in \mathcal{E}$, α_c is constant, i.e. $A \in \mathbf{R}_c$.

2.1* Proposition. *Let $\mathcal{A}$ be an $\mathcal{M}$-coreflective subcategory of $\mathcal{K}$. Then $\mathcal{A}$ determines an $(\operatorname{Iso}\mathcal{K}, \mathcal{M})$-quasi-radical J_r in $\mathcal{K}$ with $\mathbf{R}_r = \operatorname{Ob}\mathcal{A}$ and $\mathbf{S}_r = \delta_{\mathcal{M}}(\operatorname{Ob}\mathcal{A})$.*

At this stage we might just point out that if $\mathcal{K}$ is any of the categories *Rng*, *Arng* or *Grp*, a radical class $\mathbf{R}$, considered as a full subcategory of $\mathcal{K}$, need not be (normal mono)-coreflective. If, however, $\mathbf{R}$ is a strict radical class (cf. [**11**]), then the above is true. Likewise, if $\mathbf{S}$ is a semisimple class in $\mathcal{K}$, considered as a full subcategory of $\mathcal{K}$, $\mathbf{S}$ need not be (normal epi)-reflective. This is true if $\mathbf{S}$ is a strongly hereditary semisimple class. In the categories *S-act*, *Top* and *Graph* any disconnectedness $\mathbf{D}$ can be considered as an (extremal epi)-reflective subcategory – the reflection is given by the maximal $\mathbf{D}$-image.

2.2 *Definition.* An $(\mathcal{E},\overline{\mathcal{M}})$-radical in $\mathcal{K}$ is a pair (J_r, J_c) where J_r is an $(\mathrm{Iso}\,\mathcal{K},\overline{\mathcal{M}})$-quasi-radical and J_c is an $(\mathcal{E},\mathrm{Iso}\,\mathcal{K})$-quasi-coradical for which $\mathbf{R}_c=\mathbf{R}_r$ and $\mathbf{S}_c=\mathbf{S}_r$ hold. In such a case, $\mathbf{R}=\mathbf{R}_r=\mathbf{R}_c$ is called the radical-class and $\mathbf{S}=\mathbf{S}_r=\mathbf{S}_c$ the semisimple class of (J_r, J_c).

2.3 *Examples.* 1. Examples 1 and 2 in 1.2 yield $(\mathcal{E},\mathcal{M})$-radicals.

2. Examples 3, 4 and 7 in 1.2 yield $(\mathcal{E},\overline{\mathcal{M}})$-radicals.

The next theorem motivates our terminology.

2.4 Theorem. *Let (J_r, J_c) be an $(\mathcal{E},\overline{\mathcal{M}})$-radical in $\mathcal{K}$. Then $\mathbf{R}$ is an $(\mathcal{E},\mathcal{M})$-radical class and $\mathbf{S}$ is an $(\mathcal{E},\mathcal{M})$-semisimple class.*

Proof. Firstly we show that $\mathbf{R}$ is an $(\mathcal{E},\mathcal{M})$-radical class. Let $A\in\mathbf{R}$ and $\varphi: A\to B$ with $\varphi\in\mathcal{E}$ and $B\notin\mathbf{T}$. Because $\mathbf{R}_c$ and hence $\mathbf{R}$ is $\mathcal{E}$-cohereditary, $B\in\mathbf{R}$ follows. Then $1_B: B\to B$ with $1_B\in\mathcal{M}$, $B\notin\mathbf{T}$ and $B\in\mathbf{R}$ yields the first part. Suppose $A\notin\mathbf{R}$. Then $J_cA=\alpha_c: A\to A_c$ is a morphism in $\mathcal{E}$ with $A_c\notin\mathbf{T}$. Let $\psi: B\to A_c$ with $\psi\in\mathcal{M}$ and $B\notin\mathbf{T}$. If $B\in\mathbf{R}$, then $\beta_r\in\mathrm{Iso}\,\mathcal{K}$ for all $\beta_r\in J_rB$. Also, $J_cB=\beta_c$ is constant. Furthermore, because $J_cA_c\in\mathrm{Iso}\,\mathcal{K}$, $A_c\in\mathbf{S}$ from which γ_r constant follows for all $\gamma_r\in J_rA_c$. Furthermore, for any $\beta_r\in J_rB$, there exists a $\gamma_r\in J_rA_c$ and a morphism ψ_r such that $\psi_r\gamma_r=\beta_r\psi$. Hence γ is a constant monomorphism which implies $B\in\mathbf{T}$. This, however, contradicts $B\notin\mathbf{T}$. Thus $B\notin\mathbf{R}$ and $\mathbf{R}$ an $(\mathcal{E},\mathcal{M})$-radical class follows. Lastly, we show $\mathbf{S}$ is an $(\mathcal{E},\mathcal{M})$-semisimple class. Let $A\in\mathbf{S}$ and $\mu: B\to A$ with $\mu\in\mathcal{M}$ and $B\notin T$. Because $\mathbf{S}_r$ and hence $\mathbf{S}$, is $\mathcal{M}$-hereditary, $B\in\mathbf{S}$ follows. Then $1_B: B\to B$ with $1_b\in\mathcal{E}$, $B\notin\mathbf{T}$ and $B\in\mathbf{S}$ yields the first part. Suppose $A\notin\mathbf{S}$. Hence there is an $\alpha_r: A_r\to A$, $\alpha_r\in J_rA$, which is not constant. Thus $A_r\notin\mathbf{T}$, $\alpha_r\in\mathcal{M}$ and $A_r\in\mathbf{R}_r=\mathbf{R}$ holds because J_r is an $(\mathrm{Iso}\,\mathcal{K},\overline{\mathcal{M}})$-quasi-radical. Furthermore, we note that J_cA_r is constant. Let $\varphi: A_r\to B$ with $\varphi\in\mathcal{E}$ and $B\notin\mathbf{T}$. If $B\in\mathbf{S}$, then J_cB is an isomorphism and by definition there is a morphism φ_c such that $\varphi\cdot J_cB=J_cA\cdot\varphi_c$. Hence φ a constant epimorphism which contradicts $B\notin\mathbf{T}$. Thus $B\notin\mathbf{S}$ and $\mathbf{S}$ an $(\mathcal{E},\mathcal{M})$-semisimple class follows.

The next two propositions are easy consequences of the definitions.

2.5 Proposition. *If (J_r, J_c) is an $(\mathcal{E},\overline{\mathcal{M}})$-radical in $\mathcal{K}$, then $\mathbf{R}=\mathcal{R}_{\mathcal{E}}\mathbf{S}$ and $\mathbf{S}=\delta_{\mathcal{M}}\mathbf{R}$.*

2.6 Proposition. *Let (J_r, J_c) be an $(\mathcal{E},\overline{\mathcal{M}})$-radical in $\mathcal{K}$. Then:* (i) *$\mathbf{R}$ is $\mathcal{E}$-cohereditary;* (ii) *For every object A, if $\gamma: B\to A$ with $\psi\in\mathcal{M}$ and $B\in\mathbf{R}$, then* *there is an $\alpha_r \in J_rA$, $\alpha_r : A_r \to A$, such that $(B,\psi) \leq (A_r, \alpha_r)$;* (iii) *$A_c \in \mathbf{S}$ for all $A \in \mathrm{Ob}\,\mathcal{K}$ (remember $J_cA = \alpha_c : A \to A_c$).* (i)* *$\mathbf{S}$ is $\mathcal{M}$-hereditary.* (ii)* *For every object A, the quotient object (α_c, A_c) contains all quotient objects $\varphi : A \to B$ with $\varphi \in \mathcal{E}$ and $B \in \mathbf{S}$.* (iii)* *$A_r \in \mathbf{R}$ for all $\alpha_r \in J_rA$ and $A \in \mathrm{Ob}\,\mathcal{K}$.*

Lastly, in this section we turn our attention to the construction of $(\mathcal{E},\overline{\mathcal{M}})$-radicals.

2.7 Proposition. *Let $\mathbf{A}$ be an $(\mathcal{E},\mathcal{M})$-semisimple class in $\mathcal{K}$ which satisfies the following conditions:*

(i) *For every object A in $\mathcal{K}$ there is a totally unordered $\mathcal{M}$-sink $(\alpha_i : A \to A)_I$ of A with $A_i \in \mathcal{R}_{\mathcal{E}}\mathbf{A}$ such that whenever $\psi : B \to A$ with $\psi \in \mathcal{M}$ and $B \in \mathcal{R}_{\mathcal{E}}\mathbf{A}$, then $(B,\psi) \leq (A_{i_0}, \alpha_{i_0})$ holds for some $i_0 \in I$.*

(ii) *For any $\mu : A \to B$, $\mu \in \mathcal{M}$ $\alpha_i\mu \in \mathcal{M}$ holds for all $i \in I$.*

(iii) *For every object A, there is a quotient object (α_c, A_c) of A with $\alpha_c \in E$ and $A_c \in A$ which contains all other quotient objects (φ, B) of A with $\varphi \in E$ and $B \in \mathbf{A}$.*

(iv) *For any $\gamma : B \to A$, $\gamma \in \mathcal{E}$, $\gamma_{\alpha_c} \in E$ holds.*

Then $\mathbf{A}$ determines an $(\mathcal{E},\overline{\mathcal{M}})$-radical class $\mathbf{R} = \mathcal{R}_{\mathcal{E}}A$ and semisimple clas $\mathbf{S} = \mathbf{A}$.

Proof. For every $A \in \mathrm{Ob}\,\mathcal{K}$, let $J_rA = (\alpha_i : A_i \to A)_I$ the $\mathcal{M}$-sink given in (i), and let $J_cA = \alpha_c : A \to A_c$, the $\mathcal{E}$-source given in (iii). Then J_r is an $(\mathrm{Iso}\,\mathcal{K},\overline{\mathcal{M}})$-quasi-radical and J_c is an $\mathcal{E},\mathrm{Iso}\,\mathcal{K})$-quasi-coradical. We only give the proof for the first result. Let $\psi : A \to B$ with $\psi \in \mathcal{M}$ and consider $\alpha_i \in J_rA$. By (ii) $\alpha_r\psi \in \mathcal{M}$ holds and by (i), $(A_r, \alpha_r\psi) \leq (B_j, \beta_j)$ for some $\beta_j \in J_rB$. Hence condition (i) in Definition 1.1 is satisfied. In proving the second condition, we let $a_r : A_r \to A$ be arbitrary from J_rA and let $\gamma \in J_rA_r$, say $\gamma : B \to A_r$. Because $1_{A_r} : A_r \to A_r$ with $1_{A_r} \in \mathcal{M}$ and $A_r \in \mathcal{R}_{\mathcal{E}}\mathbf{A}$ holds, our hypothesis assures us the existence of a subobject (B',γ') of $A_r, \gamma' \in J_rA_r$ such that $(A_r, 1_{a_R}) \leq (B',\gamma')$. Hence $(A_r, 1_{A_r}) = (B',\gamma')$. But (B,γ) is a subobject of A_r, hence $(B,\gamma) \leq (A_r, 1_{A_r} \leq (B',\gamma')$. Because the $\mathcal{M}$-sink J_rA_r is totally unordered, $(B,\gamma) = (B',\gamma') = (A_r, 1_{A_r})$ follows which shows that γ is an isomorphism. Hence J_r an $(\mathrm{Iso}\,\mathcal{K},\overline{\mathcal{M}})$-quasi-radical follows. Lastly we show $\mathbf{R}_r = \mathcal{R}_{\mathcal{E}}\mathbf{A} = \mathbf{R}_c$. Let $A \in \mathbf{R}_r$. Then $\alpha_r \in \mathrm{Iso}\,\mathcal{K}$ for all $\alpha_r \in J_rA$. Let $\varphi : A \to B$ with $\varphi \in \mathcal{E}$ and $B \notin \mathbf{T}$. Now $\alpha_r\varphi \in \mathcal{E}$ and because $A_r \in \mathcal{R}_{\mathcal{E}}\mathbf{A}$, $B \notin \mathbf{A}$ follows. Hence $A \in \mathcal{R}_{\mathcal{E}}\mathbf{A}$. If $A \in \mathcal{R}_{\mathcal{E}}\mathbf{A}$, let $\alpha_r : J_r \to A$. Because $A \in \mathcal{R}_{\mathcal{E}}\mathbf{A}$, by (ii) there exists an $\alpha'_r : A'_r \to A$ with $\alpha'_r \in J_rA$ such that $(A, 1_A) \leq (A'_r, \alpha'_r)$. Hence $(A_r, \alpha_r) \leq (A, 1_A) = (A'_r, \alpha'_r)$ from which $(A_r, \alpha) = (A'_r, \alpha'_r) = (A, 1_A)$ follows. Thus $\alpha_r \in \mathrm{Iso}\,\mathcal{K}$ and $A \in \mathbf{R}_r$ holds. If $A \in \mathcal{R}_{\mathcal{E}}\mathbf{A}$, then $J_cA = \alpha_c : A \to A_c$ is constant. Thus $A \in \mathbf{R}_c$. Lastly, if $A \in \mathcal{R}_{\mathcal{E}}\mathbf{A}$, then there is a $\varphi : A \to B$, $\varphi \in \mathcal{E}$, $B \notin \mathbf{T}$ and $B \in \mathbf{A}$. Hence φ and then also J_cA, is not constant. This means $A \notin \mathbf{R}_c$ and $\mathbf{R}_r = \mathcal{R}_{\mathcal{E}}\mathbf{A} = \mathbf{R}_c$ follows. Because $\mathbf{A}$ is an $(,\mathcal{M})$-semisimple class, $\mathbf{A} = \delta_{\mathcal{M}}\mathcal{R}_{\mathcal{E}}\mathbf{A}$ and $\mathbf{S}_r = \mathbf{A} = \mathbf{S}_c$. follows as in the above.

Remark. If condition (i) in the above position is such that the $\mathcal{M}$-sink assigned to every object A consists of exactly one morphism, then the proposition yields an $(\mathcal{E}, \mathcal{M})$-radical.

2.8 PROPOSITION. *Let $\mathcal{A}$ be an $\mathcal{E}$-reflective subcategory of $\mathcal{K}$ such that $\mathcal{A}'$ is an $\mathcal{M}$-coreflective subcategory where $\mathcal{A}'$ is the full subcategory of $\mathcal{K}$ with* $\mathrm{Ob}\,\mathcal{A}' = \mathcal{R}_{\mathcal{E}}\mathrm{Ob}\,\mathcal{A}$. *If* $\mathrm{Ob}\,\mathcal{A}$ *is an $(\mathcal{E}, \mathcal{M})$-semisimple class, then $\mathcal{A}$ determines an $(\mathcal{E}, \mathcal{M})$-radical (J_r, J_c) with* $\mathbf{R} = \mathrm{Ob}\,\mathcal{A}'$ *and* $\mathbf{S} = \mathrm{Ob}\,\mathcal{A}$.

Proof. Follows from Propositions 2.1 and 2.1* and because $\mathrm{Ob}\,\mathcal{A} = \delta_{\mathcal{M}}\,\mathcal{R}_{\mathcal{E}}(\mathrm{Ob}\,\mathcal{A})$.

2.8* PROPOSITION. *Let $\mathcal{A}$ be an $\mathcal{M}$-coreflective subcategory of $\mathcal{K}$ such that $\mathcal{A}'$ is an $\mathcal{E}$-reflective subcategory where $\mathcal{A}'$ is the full subcategory of $\mathcal{K}$ with* $\mathrm{Ob}\,\mathcal{A}' = \delta_{\mathcal{M}}\mathrm{Ob}\,\mathcal{A}$. *If* $\mathrm{Ob}\,\mathcal{A}$ *is an $(\mathcal{E}, \mathcal{M})$-radical class then $\mathcal{A}$ determines an $(\mathcal{E}, \mathcal{M})$-radical (J_r, J_c) with* $\mathbf{R} = \mathrm{Ob}\,\mathcal{A}$, *and* $\mathbf{S} = \mathrm{Ob}\,\mathcal{A}'$.

3. $(\mathcal{E}, \overline{\mathcal{M}})$-radicals in categories subjected to certain conditions

In this section we strongly rely on the conditions, definitions and result in [**15**]. The notions constant pair, fiber, cokernel, right precise and right straight have been defined in [**14**] but can also be found in [**15**]. Firstly, we suppose $\mathcal{E}$ and $\mathcal{M}' \subseteq \mathrm{Mono}\,\mathcal{K}$ are such that $\mathcal{K}$ is an $(\mathcal{E}, \mathcal{M}')$-category (cf. [**6**] §33.) Let $\mathcal{M}$ be the class defined by $\mathcal{M} = \{\alpha \mid \alpha$ is a fiber of a morphism in $\mathcal{K}\}$. We suppose $\mathcal{K}$ satisfies the following conditions:

(A1) $\mathcal{K}$ has cokernels.

(A2) $\mathcal{K}$ has fibers.

(A3) If $\alpha\beta$ is constant and α is an epimorphism, then β is constant.

(A4) The class $\mathbf{T}$ exists in $\mathcal{K}$ and for every object A in $\mathcal{K}$ there is a T and T' in $\mathbf{T}$ and morphisms $T \to A$ and $A \to T'$.

(A5) If $\alpha\beta$ is the $(\mathcal{E}, \mathcal{M}')$-factorization of $\tau\nu$ where $\tau \in \mathcal{M}$ and $\nu \in \mathcal{E}$, then $\beta \in \mathcal{M}$ must hold.

(A6) $\mathcal{K}$ is $\mathcal{M}$-well-powered.

(A7) $\mathcal{K}$ has products.

(A8) $\mathcal{K}$ is $\mathcal{E}$-co-(well-powered).

The class of all extremal epimorphism is contained in $\mathcal{E}$ and $\mathcal{M} \subseteq \mathcal{M}'$. Using Proposition 3.6 in [**15**], it follows that for every $(\mathcal{E}, \mathcal{M})$-semi-simple class $\mathbf{S}$ in $\mathcal{K}$ and for every $A \in \mathrm{Ob}\,\mathcal{K}$, there is an extremal epimorphism $\alpha : A \to B$ with $B \in \mathbf{S}$ which contains every quotient object (γ, C) with $\gamma \in \mathcal{E}$ and $C \in \mathbf{S}$. By the previous remark, $\alpha \in \mathcal{E}$ holds. α is called the maximal $\mathbf{S}$-*image* of A in $\mathbf{S}$. $\mathbf{S}$ is called *right straight* if $F \in \mathcal{R}_{\mathcal{E}}\mathbf{S}$ for any fiber $F \to A$ of the maximal $\mathbf{S}$-image of A.

A sequence $(F_i \overset{\mu_i}{\to} A \overset{\alpha}{\to} B)_I$ in $\mathcal{K}$ is caled a *short exact sequence* if $(\mu_i : F_i \to A)_I$ is the sink of all the fibers of α and if α is the cokernel of this sink. Lastly, $\mathcal{K}$ is said to be right precise if every extremal epimorphism is the cokernel of the sink of its fibers.

3.1 THEOREM *Suppose $\mathcal{K}$ is right precise. Let (J_r, J_c) be are $(\mathcal{E}, \overline{\mathcal{M}})$-radical in $\mathcal{K}$ such that $\mathbf{S}$ is right straight. Then $(A_i \xrightarrow{\alpha_i} A \xrightarrow{\alpha_c} A_c)_I$ is a short exact sequence for every $A \in \operatorname{Ob}\mathcal{K}$ where $J_rA = (\alpha_i : A_i \to A)_I$ and $J_cA = \alpha : A \to A_c$.*

Proof. Our first observation is that $\alpha_c : A \to A_c$ is the maximal $\mathbf{S}$-image of A for every object A. Next we show that $\alpha_i\alpha_c$ is constant for each $i \in I$. Indeed, by considering the $(\mathcal{E}, \mathcal{M}')$-factorization of $\alpha_i\alpha_c$ and using Proposition 2.6 in conjuction with (A5), the desired result is obtained. We now proceed to show $\alpha_i : A_i \to A$ is a fiber of $\alpha_c : A \to A_c$. Because $\alpha_i\alpha_c$ is constant, by (A2) there is a fiber $\mu_i : F \to A$ of α_c such that $(A_i, \alpha_i) = (F, \mu)$. Because $\mathbf{S}$ is right straight, $F \in \mathcal{R}_{\mathcal{E}}\mathbf{S}$ holds. Using proposition 2.6 (ii), we can find a $j \in I$ such that $(F, \mu) \leq (A_j, \alpha_j)$. Hence $(A_i, \alpha_i) = (F, \mu) \leq (A_j, \alpha_j)$ from which $(A_i, \alpha_i) = (A_j, \alpha_j)$ follows. But then $(A_i, \alpha_i) = (F, \mu)$, i.e. $\alpha_i : A_i \to A$ is a fiber of α_c. We now show $(\alpha_i : A_i \to A)_I$ is the sink of all the fibers of α_c. Let $\mu : F \to A$ be any fiber of α_c. As above, $F \in \mathcal{R}_{\mathcal{E}}\mathbf{S}$ holds and therefore there is an $i \in I$ such that $(F, \mu) \leq (A_i, \alpha_i)$. Because $\alpha_i\alpha_c$ is constant $\alpha_i\alpha_c$ and $\mu\alpha_c$ a constant pair follows. Because μ is a fiber of $\alpha_c(A_i, \alpha_i) \leq (F, \mu)$ follows. Hence $(F, \mu) = (A_i, \alpha_i)$. Lastly, because α_c is an extremal epimorphism it is the cokernel of the sink of all its fibers because $\mathcal{K}$ is right precise. Hence $(A_i \xrightarrow{\alpha_i} A\alpha_c \xrightarrow{\alpha_c} A_c)_I$ a short exact sequence follows.

Remark. If $\mathcal{K}$ has a zero object, then each morphism has a unique fiber (more commonly known as the kernel). For an $(\mathcal{E}, \mathcal{M})$-radical in $\mathcal{K}$ with radical class $\mathbf{R}$, the short exact sequence $A_r \xrightarrow{\alpha_r} A \xrightarrow{\alpha_c} A_c$ the well known short exact sequence $\mathbf{R}(A) \to A \to A/\mathbf{R}(A)$ where $\mathbf{R}(A)$ is the embedding of the $\mathbf{R}$-radical $\mathbf{R}(A)$ of A in A.

3.2 PROPOSITION. *Let J_r be an $(\operatorname{Iso}\mathcal{K}, \overline{\mathcal{M}})$-quasi-radical and J_c an $(\mathcal{E}, \operatorname{Iso}\mathcal{K})$-quasi-coradical in $\mathcal{K}$ such that for all objects A, $(A_i \xrightarrow{\alpha_i} A \xrightarrow{\alpha_c} A_c)_I$ is a short exact sequence where $J_rA = (\alpha_i : A_i \to A)$, and $J_cA = \alpha_c$. Then (J_r, J_c) is an $(\mathcal{E}, \overline{\mathcal{M}})$-radical in $\mathcal{K}$.*

Proof. We have to show $\mathbf{S}_c = \mathbf{S}_r$ and $\mathbf{R}_c = \mathbf{R}_r$. These equalities are obvious from the following more general results: Let $(B_i \xrightarrow{\mu_i} C \xrightarrow{\gamma} D)_I$ be any short exact sequence in $\mathcal{K}$. Then γ is an isomorphism if and only if μ_i is constant for each $i \in I$. Secondly, γ is constant if and only if μ_i is an isomorphism for each $i \in I$.

Lastly we give the relation between $(\mathcal{E}, \mathcal{M})$-radicals and radical functors. For this purpose we retain all the conditions imposed on $\mathcal{K}$ and add another, i.e., we suppose $\mathcal{K}$ has a zero object. Then $\mathcal{M}$ becomes the class of all normal monomorphisms (kernels). We also choose our $\mathcal{E}$ and $\mathcal{M}'$ more specifically. Let $\mathcal{E}$ be the class of all normal epimorphisms and let $\mathcal{M}'$ be the class of all monomorphisms in $\mathcal{K}$. In such a category, it is well known that $\mathbf{A} \subseteq \operatorname{Ob}\mathcal{K}$ is an $(\mathcal{E}, \mathcal{M})$-radical class if and only if the following three conditions are satisfied:

(R1) $\mathbf{A}$ is $\mathcal{E}$-cohereditary

(R2) For every object A there is a normal subobject (A_r, α_A) of A, with $A_r \in \mathbf{A}$, which contains every other normal subobject (B, μ) of A with $B \in \mathbf{A}$. This subobject (A_r, α_A) of A is called the $\mathbf{A}$-radical of A.

(R3) If $\beta_A : A \to A_c$ is the cokernel of $\alpha_A : A_r \to A$, then $(A_c)_r = 0$ (i.e. the A-radical of A_c is the zero subobject).

An $(\mathcal{E}, \mathcal{M})$-radical class has the ADS-property (cf. [**15**]) if $\alpha_A \mu \in \mathcal{M}$ for any $\mu : A \to B$, $\mu \in \mathcal{M}$ and every object A in $\mathcal{K}$.

3.3 Proposition. *Let A be an $(\mathcal{E}, \mathcal{M})$-semisimple class such that $\mathcal{R}_{\mathcal{E}}\mathbf{A}$ satisfies the ADS-property. Then $\mathbf{A}$ determines an $(\mathcal{E}, \mathcal{M})$-radical (J_r, J_c) with radical class $\mathcal{R}_{\mathcal{E}}\mathbf{A}$ and semisimple class $\mathbf{A}$.*

Proof. In view of the above characterization of an $(\mathcal{E}, \mathcal{M})$-radical class, the ADS-property on $\mathcal{R}_{\mathcal{E}}\mathbf{A}$ and the fact that the composttion of two normal epimorphisms in is normal again, we only have to prove condition (iii) in Proposition 2.7. Let $A \in \operatorname{Ob}\mathcal{K}$ with $\beta_A : A \to A_c$ the cokernel of $\alpha_A : A_r \to A$ where (A_r, α_A) is the $\mathcal{R}_{\mathcal{E}}\mathbf{A}$-radical of A. Let $\varphi : A \to B$ be a morphism with $\varphi \in \mathcal{E}$ and $B \in \mathbf{A}$. Consider the $(\mathcal{E}, \mathcal{M}')$-factorization of $\alpha_A\varphi$, say $A_r \xrightarrow{\alpha_A\varphi} B = A_r \xrightarrow{\tau} D \xrightarrow{\nu} B$. Using (A5), $\nu \in \mathcal{E}$ follows. Hence $D \in \mathcal{R}_{\mathcal{E}}\mathbf{A} \cap \mathbf{A} = \mathcal{R}_{\mathcal{E}}\mathbf{A} \cap \delta_{\mathcal{M}}\mathcal{R}_{\mathcal{E}}\mathbf{A}$ which makes D the zero object. Then, because $\alpha_A\varphi$ is constant, the definition of a normal epimorphism yields a unique $\eta : A_r \to B$ such that $\beta_A\eta = \varphi$. Hence (β_A, A_c) is the desired quotient object because $A_c \in \mathbf{A}$ by (R3) above.

3.4 Proposition. *Let $\mathbf{B}$ be an $(\mathcal{E}, \mathcal{M})$-radical class whiclt satisfies the ADS-property. Then $\mathbf{B}$ determines an $(\mathcal{E}, \mathcal{M})$-radical with radical class $\mathbf{B}$ and semisimple class $\delta_{\mathcal{M}}\mathbf{B}$.*

We recall the following:

1. Let F_1 and F_2 be functors from a category $\mathcal{A}$ into a category $\mathcal{B}$. Then F_1 is called a (*normal*) *subfunctor* of F_2 if the followin conditions are satisfied:

$$\begin{tikzcd} F_1 A \arrow[r, "F_1\gamma"] \arrow[d, "A"'] & F_1 B \arrow[d, "B"] \\ F_2 A \arrow[r, "F_2\gamma"'] & F_2 B \end{tikzcd}$$

(i) For every $A \in \operatorname{Ob}\mathcal{A}$, $F_1 A$ is a (normal) subobject of $F_2 A$. (Let us indicate this monomorphism by $\eta_A : F_1 A \to F_2 A$.)

(ii) For every morphism $\gamma : A \to B$ in A, the following diagram commutes:

2. Following Marki and Wiegandt [**9**], we give Carreau's definition of radical functor [**4**] in the slightly modified, but equivalent version of Holcombe and Walker [**7**]. A covariant functor $F : \mathcal{E}(\mathcal{K}) \to \mathcal{K}$ is called a radical functor if: (i) F is a normal subfunctor of the inclusion functor $I : \mathcal{E}(\mathcal{K}) \to \mathcal{K}$. (For every A, this normal subobjet of A wil be denoted by (FA, η_A).) (ii) If $\gamma_A : A \to A_F$ is the cokernel of η_A, then $FA_F = 0$.

A radical B functor F is said to be *complete* if for every normal monomorphism $\psi : B \to A$ for which $FB = B$, the inequality $(B, \psi) \leq (FA, \eta_A)$ holds. F is called *idempotent* if $F(FA) = FA$.

3.5 PROPOSITION*Any $(\mathcal{E}, \mathcal{M})$-radical uniguely determines a complete and idemotent radical functor.*

Proof. The radical class of an $(\mathcal{E}, \mathcal{M})$-radical (J_r, J_c) is uniquely determined. Using Corollary 2.12 in [**4**], any radical class $\mathbf{R}$ determines a complete and idempotent radical functor F defined by $FA = A_r$ with $\eta_A = J_r A$.

The converse is given by using the next well known result (cf. for example [**7**]).

3.6 PROPOSITION. *Let F be a complete and idempotent radical functor. Let $\mathbf{R} = \{A \in \mathrm{Ob}\,\mathcal{K} | \, FA = A\}$. Then $\mathbf{R}$ is an $(\mathcal{E}, \mathcal{M})$-radical class and $\delta_{\mathcal{M}}\mathbf{R} = \{A \in \mathrm{Ob}\,\mathcal{K} | \, FA = 0\}$.*

3.7 COROLLARY. If, in the notation of 3.6, $\mathbf{R}$ satisfies the ADS-property, the F determines an $(\mathcal{E}, \mathcal{M})$-radical with radical class $\mathbf{R} = \{A | \, FA = A\}$ and semisimple class $\mathbf{S} = \{A | \, FA = 0\}$.

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