PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 38 (52), 1985, pp. 35–38

ON THE MINIMAL DISTANCE OF THE ZEROS OF A POLYNOMIAL

Slaviša B. Prešić

1. Let

$$p(x) = \sum_{\nu=0}^{n} a_\nu x^\nu, \quad (a_\nu \in C, \ \ a_n \neq 0) \tag{1}$$

be a complex polynominal whose zeros $x_1, \dots, x_n$ are mutually distinct. In this paper we give a method of finding some positive lower bounds of

$$\min_{i \neq j} |x_i - x_j|. \tag{2}$$

2. In the sequel we shall use some well known facts about polynomials. Let $p(x) = a_0 + \cdots + a_n x^n$ $(a_n \neq 0)$ be any complex polynomial. There are many known formulas ([**1**], [**2**]) of the type

$$|x_i| \leq M \qquad (i = 1, \dots, n) \tag{3}$$

where $x_1, \dots, x_n$ are all zeros of $p(x)$ and M is a positive constant. So, a classical result due to Cauchy [**1**] is

$$|x_i| \leq 1 + \max_{1 \leq i < n} (|a_i|/|a_n|) \tag{4}$$

We emphasize that in this case, and the same is almost ever, M has the following poroperty

(5) $\qquad M$ *is an increasing function in each* $|a_0|/|a_n|, \dots, |a_{n-1}|/|a_n|$

Let, further, besides $p(x)$

$$p_1(x) = b_0 + \cdots + b_m x^m, \qquad (b_m \neq 0)$$

AMS Subject Classification (1980): Primary 12D10, 26C10, 30C15.

be another complex polynomial. Then there is a polynomial of the form

$$r(x) = c_0 + \cdots + c_{n-1}x^{n-1} \tag{6}$$

such that the equality

$$p_1(x_i) = r(x_i) \tag{7}$$

holds for every zero x_i of the polynomial $p(x)$. In other words we have the following relation

$$p_1(x) \equiv r(x) \ (\mod p(x))$$

There are at least two methods of finding $r(x)$: by the division algorithm or by applying, enough number of times, the substitution

$$x^n \to -a_n^{-1}(a_0 + a_1x + \cdots + a_{n-1}x^{n-1}).$$

Note that for $r(x)$ we shall also use the notation $r(p_1(x), p(x))$.

Suppose now that we would like to have a polynomial $r(x)$ of the form (6) such that the equality

$$1/p_1(x_i) = r(x_i) \tag{8}$$

holds for every zero x_i of $p(x)$. Generally such a polynomial $r(x)$ does not exist. It exists just in the case the polynomials $p(x)$ and ${}_1(x)$ have no common zero, i.e. they are relatively prime polynomials. Then $r(x)$ can be found by the Euclidean algorithm, for example. Namely, in such a way we can find two polynomials $e_1(x)$, $e_2(x)$ such that the identity

$$e_1(x)p(x) + e_2(x)p_1(x) = 1$$

holds. Hence we have the equality $1/p_1(x_i) = e_2(x_i)$ and consequently for $r(x)$ we may take the polynomial $r(e_2(x), p(x))$. Note that for the obtained $r(x)$, i.e. $r(e_2(x), p(x))$ we shall also use the notation as before: $r(1/p_1(x), p(x))$. More generally, if $a(x)/b(x)$ is any rational function, where $b(x)$ and $p(x)$ are relatively prime polynomials, then by $r(a(x)/b(x), p(x))$ will be denoted a polynomial of the form (6) such that the equality $a(x_i)/b(x_i) = r(x_i)$ holds for every zero x_i of the polynomial $p(x)$. Obviously the polynomial $r(x)$ a unique.

Example 1. Let

$$p(x) = x^3/3 - x^2 + 2x + 1/3, \qquad p_1(x) = x^2 - 2x + 2. \tag{9}$$

Then using the Euclidean algorithm we obtain the following polynomial equalities

$$p(x) = p_1(x)(x-1)/3 + (2x+3)/3), \qquad p_1(x) = (2x+3)/3 \cdot (3x/2 - 21/4) + 29/4$$

from which on eliminating the polynomial $(2x+3)/3$ we infer the equality

$$p(x)(-6x+42)/29 + p_1(x)(2x^2 - 9x + 11)/29 = 1$$

Thus we see that

$$r(1/p_1(x), p(x)) = (2x^2 - 9x + 11)/29 \tag{10}$$

3. Now we are going to describe, step by step, a method of finding a lower bound of (2) for a given polynomial (1).

Firstly, we begin with the Taylor formula

$$p(x) = p(x_i) + (x - x_i)p'(x_i) + \cdots + (x - x_i)^n \frac{p^{(n)}(x_i)}{n!}$$

where $i \in \{1, \ldots, n\}$ is fixed. Hence we conclude that the following equation in d

$$d^{n-1} \cdot p'(x_i) + d^{n-2} \cdot p''(x_i) + \cdots + d \cdot \frac{p^{(n-1)}(x_i)}{(n-1)!} + \frac{p^{(n)}(x_i)}{n!} = 0 \tag{11}$$

has the zeros $(x_1 - x_i)^{-1}, \ldots, (x_{i-1} - x_i)^{-1}, (x_{i+1} - x_i)^{-1}, \ldots, (x_n - x_i)^{-1}$.

Secondly, let

$$M(|p^{(n)}(x_i)/n!p'(x_i)|,\, p^{(n-1)}(x_i)/(n-1)!p'(x_i)|, \ldots, |p''(x_i)2!p'(x_i)|) \tag{12}$$

be any increasing (in the sense of (5)) upper bound of the moduli of the zeros of the equation (11). Thus, we have the inequality

$$\begin{aligned} |x_j - x_i| \le M(&|p^{(n)}(x_i)/n!p'(x_i)|,\, p^{(n-1)}(x_i)/(n-1)!p'(x_i)|, \ldots \\ &\ldots, |p''(x_i)2!p'(x_i)|) \end{aligned} \tag{13}$$

Thirdly, suppose that a constant $A > 0$ is an upper bound of $|x_i|$ $(i = 1, \ldots, n)$.

Fourthly, suppose that we have determined the following polynomials

$$r(p^{(n)}(x)/n!p'(x)), \quad r(p^{(n-1)}(x)/(n-1)!p'(x), \ldots, r(p''(x)/2!p'(x))$$

which exist since $p(x)$ has mutually distinct zeros. Denote these polynomials by

$$r_n(x),\, r_{n-1}(x), \ldots, r_2(x)$$

respectively.

For any polynomial $f(x) = f_0 + f_1 x + \ldots, + f_s x^s$ let $|f|(x)$ denote the polynomial $f_0| + |f_1|x + \cdots + |f_s|x^s$.

Fifthly, using the monotony of M and the inequalities $|x_i| \le A$ from (13) it follows that

$$|x_j - x_i| \ge (M(|r_n|(A), |r_{n-1}|(A), \ldots, |r_2|(A)))^{-1} \quad (i \ne j)$$

which yields our final result.

THEOREM. *The minimal distance of the zeros of the polynomial* (1) *satisfies the inequality*

$$\min_{j\neq i}|x_j - x_i| \geq (M(|r_n|(A), |r_{n-1}|(A), \ldots, |r_2|(A)))^{-1} \tag{14}$$

Example 2. Let $p(x)$ be the polynomial considered in Example 1. Then we have the following equalities

$$p'(x) = p_1(x), \quad p''(x) = 2x - 2, \quad p'''(x) = 2$$

As we have already established we have the equality (see (10))

$$r(1/p'(x), p(x)) = (2x^2 - 9x + 11)/29 \tag{15}$$

In the next step we should decide which the M-formula to use. Let us take the Cauchy's one. So, according to (13) we have the following inequality

$$|x_j - x_i|^{-1} \leq 1 + \max(|p''(x_i)/2p(x_i)|, |p'''(x_i)/6p'(x_i)|)$$

i.e. the inequality

$$|x_j - x_i|^{-1} \leq 1 + \max(|x_i - 1/x_i^2 - 2x_i + 2|, |1/3(x_i^2 - 2x_i + 2)|)$$

Using (15) it is easily seen that

$$\frac{1}{x^2 - 2x + 2} \equiv \frac{2x^2 - 9x + 11}{29} \quad (\mod p(x)),$$
$$\frac{x - 1}{x^2 - 2x + 2} \equiv \frac{-5x^2 + 8x - 13}{29} \quad (\mod p(x))$$

Thus the inequality of the type (14) reads

$$\min_{i\neq j}|x_j - x_i| \leq 1\Big/\left(1 + \max\left(\frac{2A^2 + 9A + 11}{29}, \frac{5A^2 + 9A + 13}{29}\right)\right) \tag{17}$$

where A is an upper bound of $|x_1|, |x_2|, |x_3|$. For instance, using the Cauchy formula (4) we conclude that

$$\min_{i\neq j}|x_j - x_i| \geq 29/350.$$

REFERENCES

[1] M. Marden, *Geometry of Polynomials*, Amer. Math. Society, Providence, Rhode Island,1966.

[2] S. Zervos, *Aspects modernes de la localisation des zéros des polynomes d'une variable*, Thèse Sc. Math., Gauthier-Villars & C$^{\text{ie}}$, Paris, 1960.

Institut za matematiku
Prirodno-matematički fakultet
11000 Beograd

(Received 03 10 1984)