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SOME REMARKS ON REPRODUCTIVE SOLUTIONS

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Abstract. Prešić considered in [**1**] the most general equations, i.e. the equations of the form $r(x) = \top$, where r is a relation on a given set S ($\top$ and $\perp$ are two **diferent** objects — truth values). We consider equations of this form and give some propositions related to the reproductive solutions.

In [**1**] Prešić described by the formula $F(x) = r(x) \cdot x + r'(x) \cdot A(h(x))$ all the reproductive solutions of the equation $r(x) = \top$, where $A : S \to S$ is a given solution of the equation $r(x) = \top$, $h : S \to S$ is a parameter, and $+$ and $\cdot$ are two operations satisfying certain rules. By (1) we describe the formulas of all reproductive solutions of the equation $r(x) = \top$.

In [**2**] Prešić gave the reproductive solution $x = B(q, J(q), J(\alpha_q(q)), \dots, J(\alpha_q^{k-2}(q)))$ of the equation $J(x) = 0$ on the set S of k elements. The cycle α_q ranges over the set S and the function B "chooses" the solution. Using this idea we give another formula of the reproductive solution on the set of k elements and the formula of the reproductive solution on an arbitrary set.

Definition 1. The formula $x = f(t)(f : S \to S)$ defines the general parametric solution or, simply, the general solution of the equation $r(x) = \top$ if and only if:

$$\forall(x) r(f(x)) = \top \wedge (\forall(x)(r(x) = \top) \Rightarrow (\exists t)(x = f(t))).$$

Definition 2. Let $x = f(t)$ be a parametric solution of the equation $r(x) = \top$. If $(\forall x)(r(x) = \top \Rightarrow x = f(x))$ then the parametric solution $x = f(t)$ is called reproductive.

Let $x = A(t)$ be the general solution of the equation $r(x) = \top$, $G : S \times \{\top, \perp\} \times R \to S$ (R is the set of solutions of the equation $r(x) = \top$, i.e. $r(x) = \top Leftrightarrow x \in R$) and $h : S \to S$ is a parameter. Then we have:

PROPOSITION 1. *The formula*

$$x = G(t, r(t), A(t)) \tag{1}$$

AMS Subject Classification (1980): Primary 04A05 *defines all general reproductive solutions of the equation* $r(x) = \top$ *if the function* $G(x, y, z)$ *is defined in the following way*

$$G(x, \top, z) = x, \quad G(x, \bot, z) = z.$$

Proof. Let us prove that (1) satisfies the equation $r(x) = \top$. If $t \in R$ then $G(t, r(t), A(h(t))) = t \in R$. If $t \in S \setminus R$, then $G(t, r(t), A(t)) = A(h(t)) \in R$. Let $x = F(t)$ be the general reproductive solution of the equation $r(x) = \top$. The solution $x = F(t)$ can be obtained from (1). If $t \in R$, then $G(t, r(t), A(h(t))) = t = F(t)$. Let $t \in S \setminus R$ and $F(t) = x \in R$. Since A is the general solution, there is a $u \in S$ such that $A(u) = x$. Let $h(t) = u$. Then $G(t, r(t), A(h(t))) = A(h(t)) = A(u) = x = F(t)$. $\square$

Let S be a set containing k elements, and let $p : S \to S$ be a function satisfying the condition

$$\{t, p(t), p^2(t), \ldots, p^{k-1}(t)\} = S, \qquad t \in S$$

where $p^0(t) = t, p^{n+1}(t) = p(p^n(t))$. The function $B : S \times \{\top, \bot\} \to S$ is defined in the following way:

$$B(x, y) = \begin{cases} x & y = \top \\ p(x), & y = \bot. \end{cases}$$

Finally, let $M(x) = B(x, r(x))$.

PROPOSITION 2. *Let* $r(x) = \top$ *be the consistent equation. Then the formula* $x = M^{k-1}(t)$ *defines the general reproductive solution of the equation* $r(x) = \top$.

Proof. Prove that $M^{k-1}(t)$ satisfies the equation $r(x) = \top$. Let $p^i(t)$ be the first element of the sequence $t, p(t), p^2(t), \ldots, p^{k-1}(t)$ which is a solution of the equation $r(x) = \top$. Then $M^{h-1}(t) = p^i(t)$, i.e. $M^{k-1}(t)$ satisfies the equation $r(x) = \top$. Let $r(x) = \top$. Putting $t = x$, we have $M^{k-1}(t) = M^{k-1}(x) = x$, by the definition of the function M.

If the equation is given in the form $I(x) = 0$, where $I : S \to E$ and E contains 0, then the function $B : S \times E \to S$ is defined as

$$B(x, y) = \begin{cases} x, & y = 0 \\ p(x), & y \neq 0. \end{cases}$$

Example. Solve the equation $ax \cup bx' = 0$ on the Boolean algebra B_2. Here $p(t) = t'$ and

$$M(t) = B(t, I(t)) = I'(t)t \cup I(t)p(t) = I(t)t \cup I(t)t'$$

i.e.

$$\begin{aligned} x = M^{2-1}(t) = M(t) = (at \cup bt')'t \cup (at \cup bt')t' = \\ = (a' \cup t')(b' \cup t)t \cup bt' = a'b't \cup a't \cup bt = a't \cup bt'. \end{aligned}$$

Let $P(S)$ be the power set of the set S. Define the function $B : S \times \{\top, \bot\} \to P(S)$ in the following way:

$$B(x,y) = \begin{cases} \{x\}, & \text{for } y = \top \\ \emptyset, & \text{for } y = \bot. \end{cases}$$

If $B(x, r(x)) = N(x)$, then we have:

PROPOSITION 3. *The set of solutions of the equation* $r(x) = \top$ *is defined by the formula* $R = \bigcup_{t \in S} N(t)$.

The proof follows from the definition of the function $N(x)$.

PROPOSITION 4. *Let* S *be a well-ordered set, and* $r(x) = \top$ *a consistent equation. The general solution of the equation* $r(x) = \top$ *is defined by*

$$x = \max(N(t) \cup \{\min \bigcup_{p \in S} N(p)\}). \tag{2}$$

This solution is reproductive.

Proof. Let $\max(N(t) \cup \{\min \bigcup_{p \in S} N(p)\}) = g(t)$. If $r(t) = \top$, then $N(t) = \{t\}$. Since $\min \bigcup_{p \in S} N(p)$ is the minimal element of the set of solutions, by Proposition 3, we have

$$\max(\{t\} \cup \{\min \bigcup_{p \in S} N(p)\}) = t \in R.$$

If $r(t) = \bot$ then $N(t) = \emptyset$. So

$$\max(\emptyset \cup \{\min \bigcup_{p \in S} N(p)\}) = \min \bigcup_{p \in S} N(t) \in R. \qquad \square$$

Remark that the function $B(x, y)$ in the last proposition can be defined as

$$B(x,y) = \begin{cases} \{x\}, & \text{for } y = \top \\ \{a\}, & \text{for } y = \bot. \end{cases}$$

where $a = \min S$.

The formula (3) defines the general (reproductive) solution of the equation $r(x) = \top$. In order to obtain all general reproductive solutions we can use the formula $x = G(t, r(t), g(h(t)))$.

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